



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

Week 6: Finite element formulation for plane trusses and frames

Lecture 27: Element Formulation

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In This Week

- Application of FEM to Plane Truss and Frame Problems
- Computer Implementation
- Concept of Error and Convergence

Steps in Finite Element Method

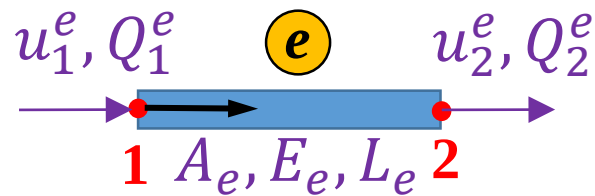
- ↓ Boundary Value Problem
- ↓ Discretization of the Domain
- ↓ Derivation of Element Equations
- ↓ Assembly of Element Equations based on Element Connectivity
- ↓ Imposition of Boundary Conditions
- ↓ Solutions
- Post-processing

Recall: Element Formulation

1D Bar Problem



Element: Building block



$\bar{y}$ $\bar{x}$ Element/ Local
coordinate

Element Equation

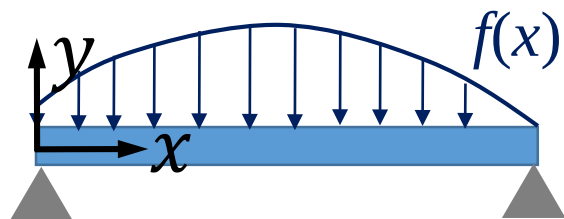
$$[K^e]\{u^e\} = \{Q^e\}$$

$$[K^e] = \frac{A_e E_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

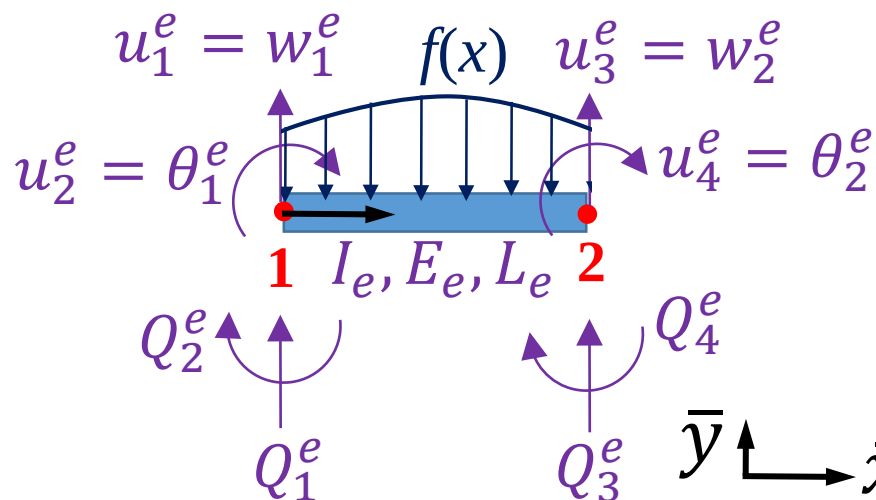
$$\{u^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} \quad \{Q^e\} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

Recall: Element Formulation

Euler-Bernoulli Beam



Element: Building block



Element Equation

$$[K^e]\{u^e\} = \{F^e\} \quad \{F^e\} = \{f^e\} + \{Q^e\}$$

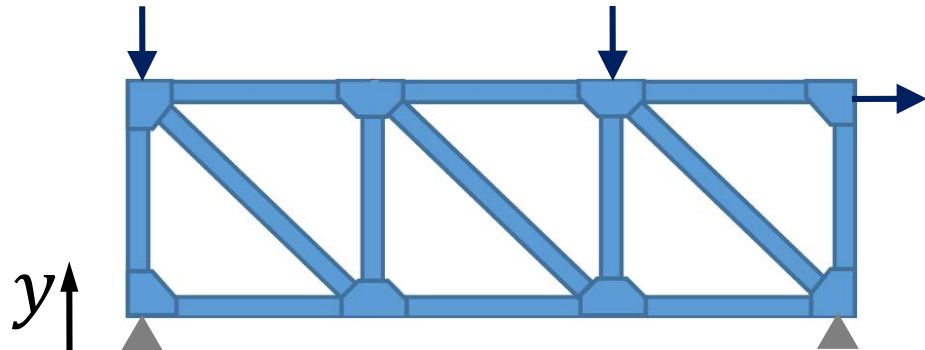
$$[K^e] = \frac{2E_e I_e}{L_e^3} \begin{bmatrix} 6 & -3L_e & -6 & -3L_e \\ -3L_e & 2L_e^2 & 3L_e & L_e^2 \\ -6 & 3L_e & 6 & 3L_e \\ -3L_e & L_e^2 & 3L_e & 2L_e^2 \end{bmatrix}$$

$$f_i^e = \int_{x_1}^{x_2} f(x) \phi_i^e dx$$

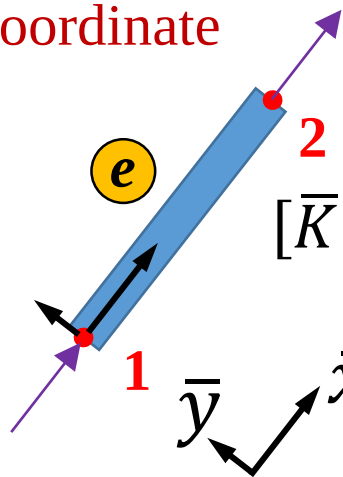
Element/ Local
coordinate

Element Formulation

Plane Truss

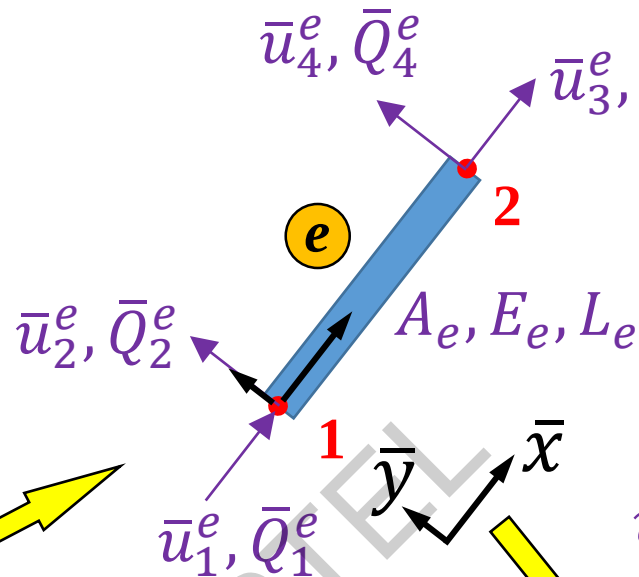


Global
coordinate

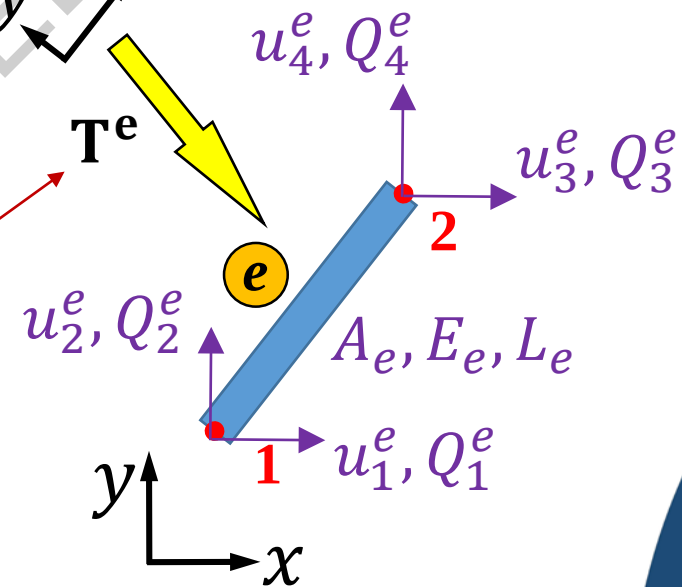


$$[\bar{K}^e] = \frac{A_e E_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element/ Local
coordinate



Local
to
Global

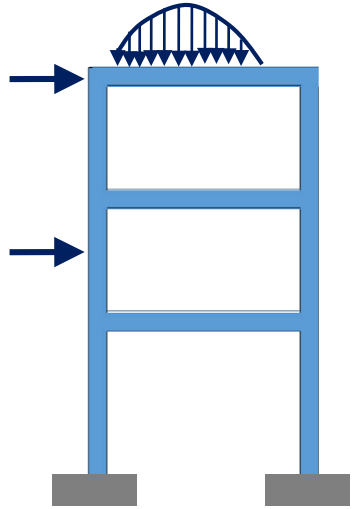
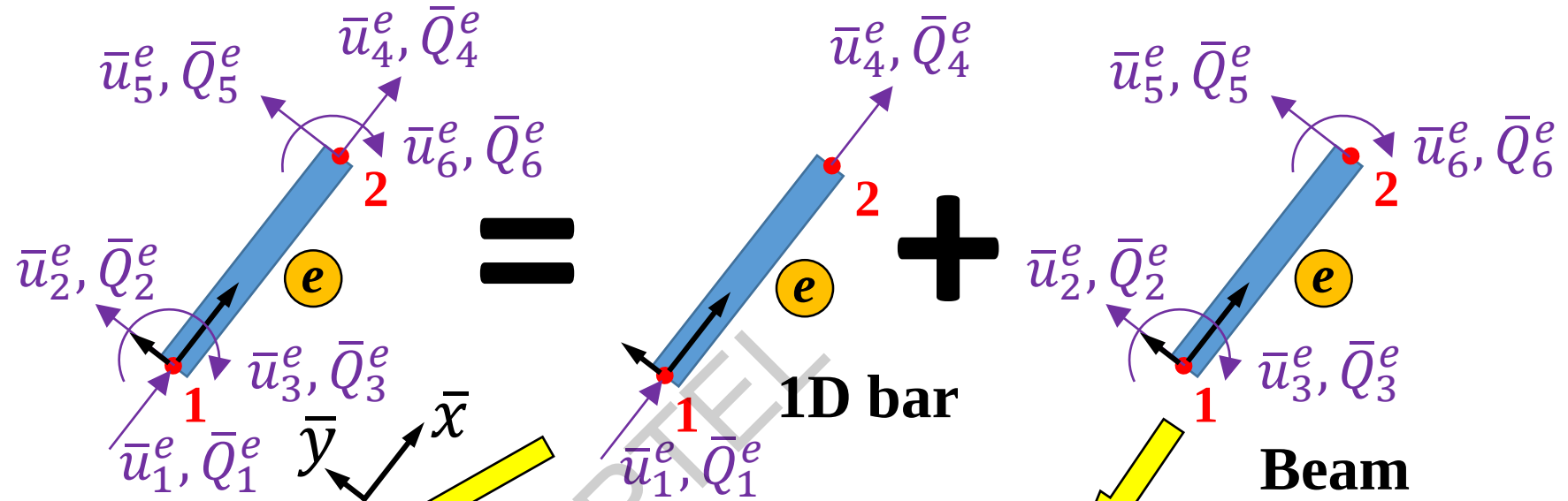


$$[\bar{K}^e] = \frac{A_e E_e}{L_e} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{K}^e] \{\bar{u}^e\} = \{\bar{Q}^e\}$$

Element Formulation

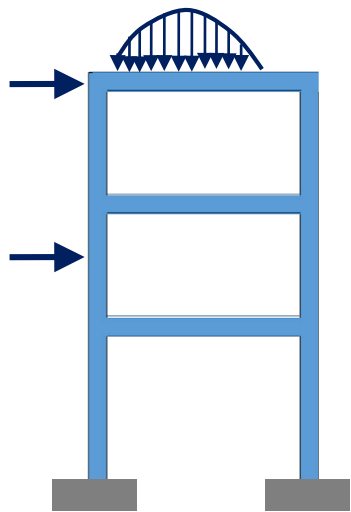
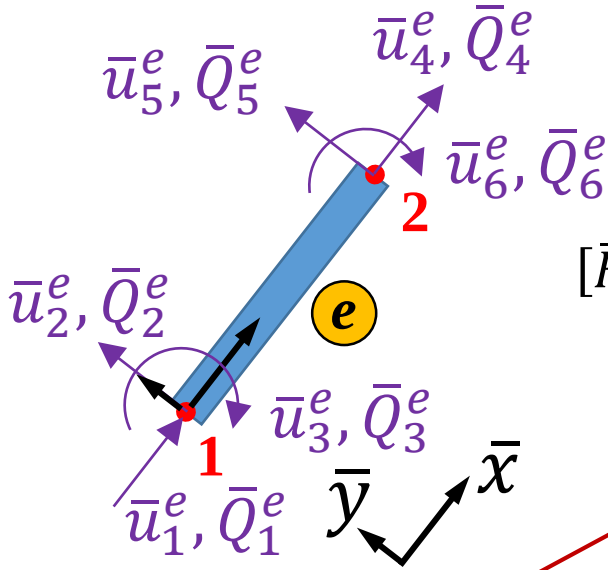
Plane Frame



$$\frac{A_e E_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 4 \end{matrix} = \frac{2E_e I_e}{L_e^3} \begin{bmatrix} 6 & -3L_e & -6 & -3L_e \\ -3L_e & 2L_e^2 & 3L_e & L_e^2 \\ -6 & 3L_e & 6 & 3L_e \\ -3L_e & L_e^2 & 3L_e & 2L_e^2 \end{bmatrix} \begin{matrix} 2 \\ 3 \\ 5 \\ 6 \end{matrix}$$

Element Formulation

Plane Frame



$$[\bar{K}^e] = \frac{2E_e I_e}{L_e^3} \begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3L_e & 0 & -6 & -3L_e \\ 0 & -3L_e & 2L_e^2 & 0 & 3L_e & L_e^2 \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3L_e & 0 & 6 & 3L_e \\ 0 & -3L_e & L_e^2 & 0 & 3L_e & 2L_e^2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

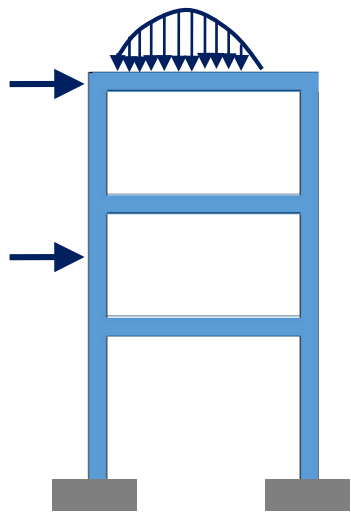
$$\mu = \frac{A_e L_e^2}{2I_e}$$

$$\frac{A_e E_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 4 \end{matrix}$$

$$\frac{2E_e I_e}{L_e^3} \begin{bmatrix} 6 & -3L_e & -6 & -3L_e \\ -3L_e & 2L_e^2 & 3L_e & L_e^2 \\ -6 & 3L_e & 6 & 3L_e \\ -3L_e & L_e^2 & 3L_e & 2L_e^2 \end{bmatrix} \begin{matrix} 2 \\ 3 \\ 5 \\ 6 \end{matrix}$$

Element Formulation

Plane Frame



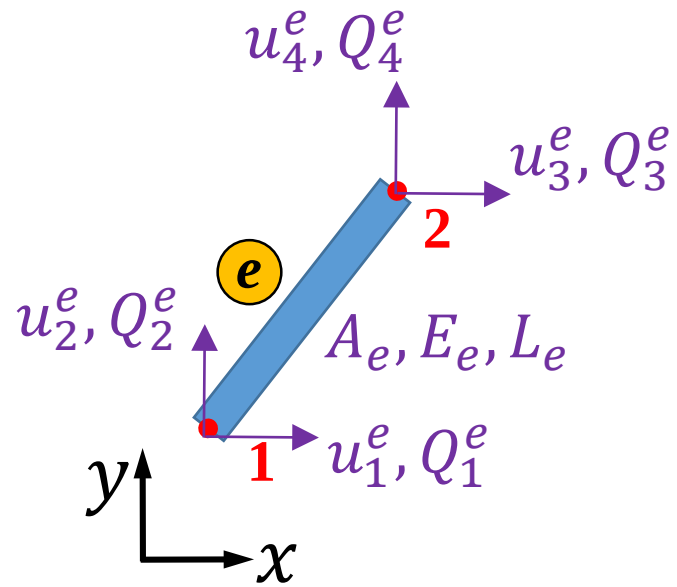
$$[\bar{K}^e] = \frac{2E_e I_e}{L_e^3} \begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3L_e & 0 & -6 & -3L_e \\ 0 & -3L_e & 2L_e^2 & 0 & 3L_e & L_e^2 \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3L_e & 0 & 6 & 3L_e \\ 0 & -3L_e & L_e^2 & 0 & 3L_e & 2L_e^2 \end{bmatrix}$$

$$\mu = \frac{A_e L_e^2}{2I_e}$$

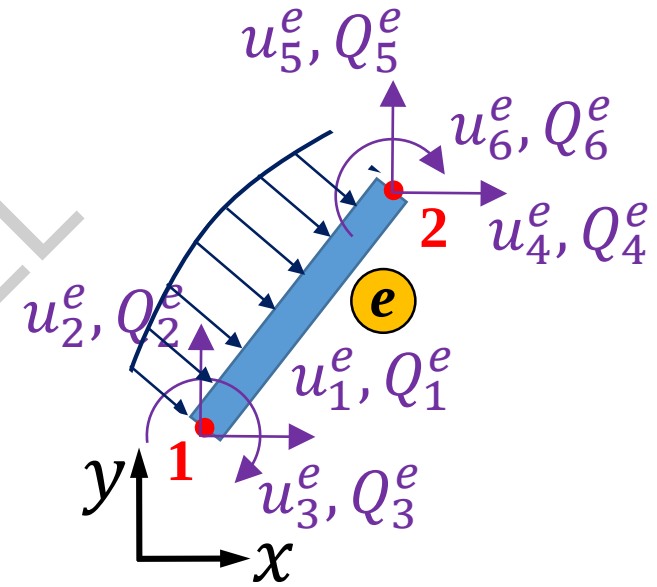
Local to Global

Element Formulation

Truss Vs. Frame



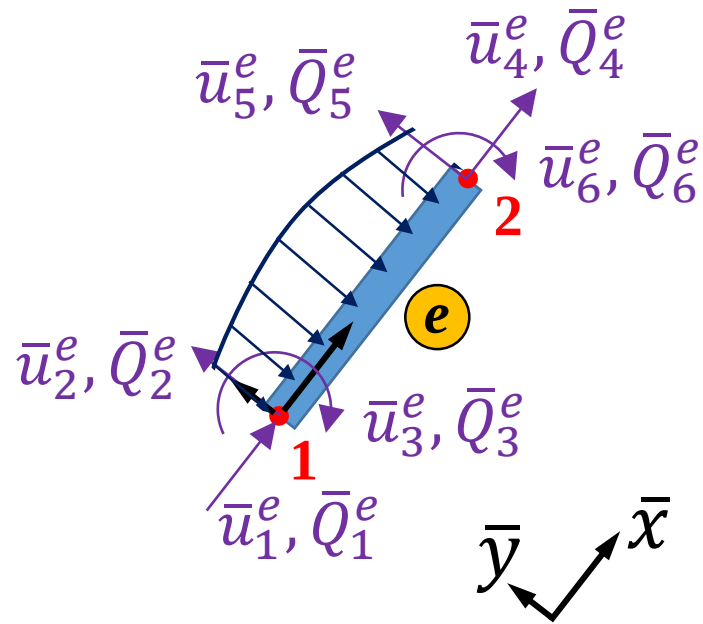
Truss



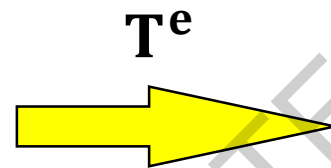
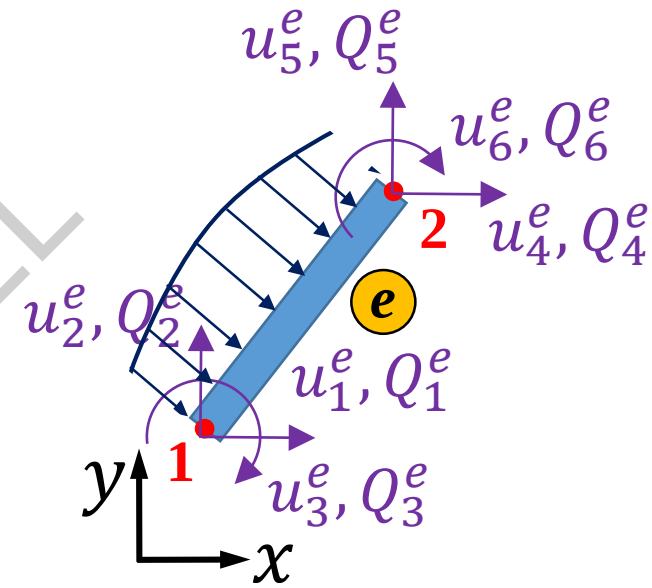
Frame

Element Formulation

Transformation from Local Coordinate to Global Coordinate

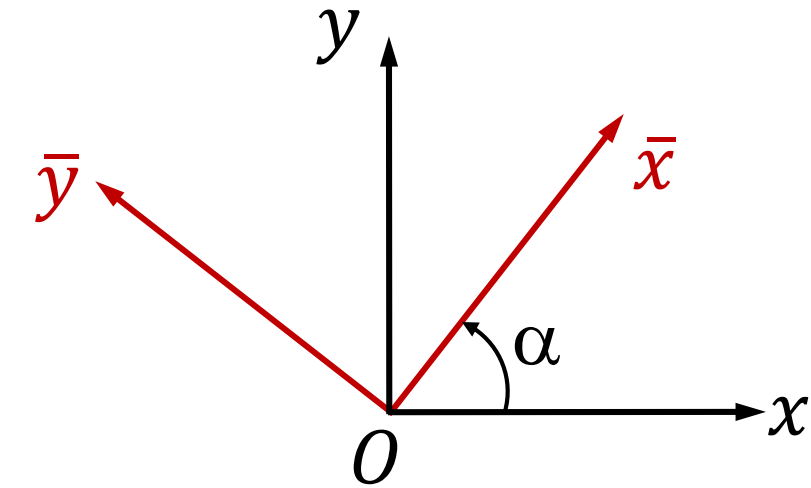


$$[\bar{K}^e]\{\bar{u}^e\} = \{\bar{F}^e\}$$

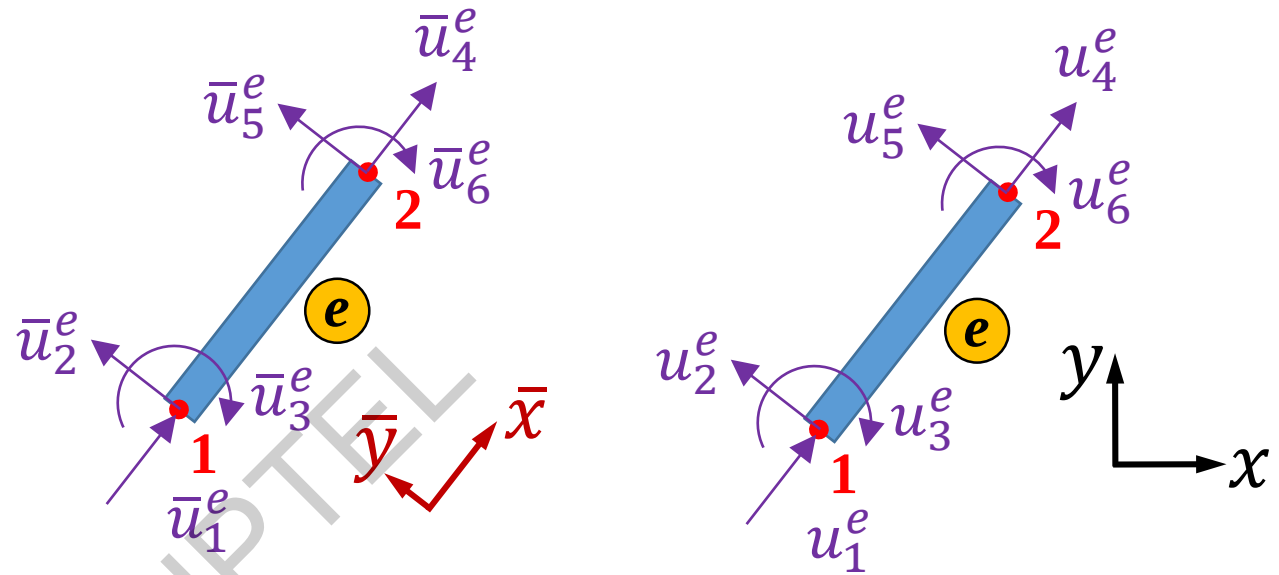


$$[K^e]\{u^e\} = \{F^e\}$$

Transformation from Local Coordinate to Global Coordinate



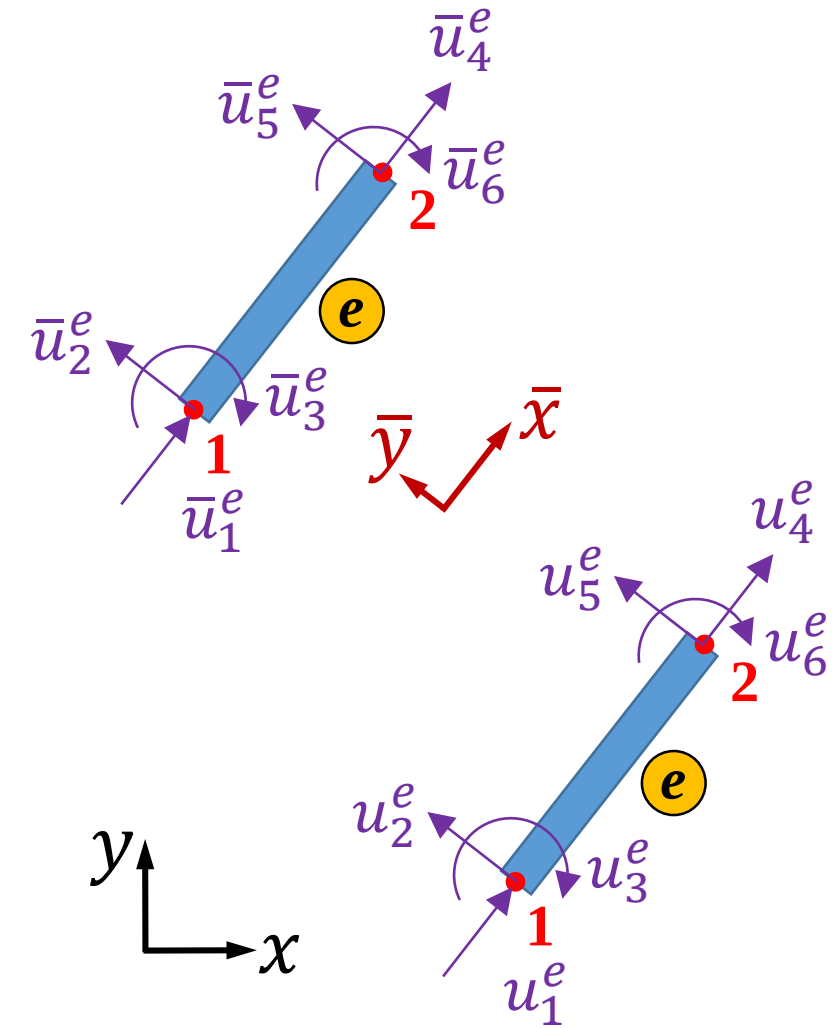
$$\begin{Bmatrix} \bar{x} \\ \bar{y} \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$



$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{u}_4^e \\ \bar{u}_5^e \\ \bar{u}_6^e \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix}$$

Transformation from Local Coordinate to Global Coordinate



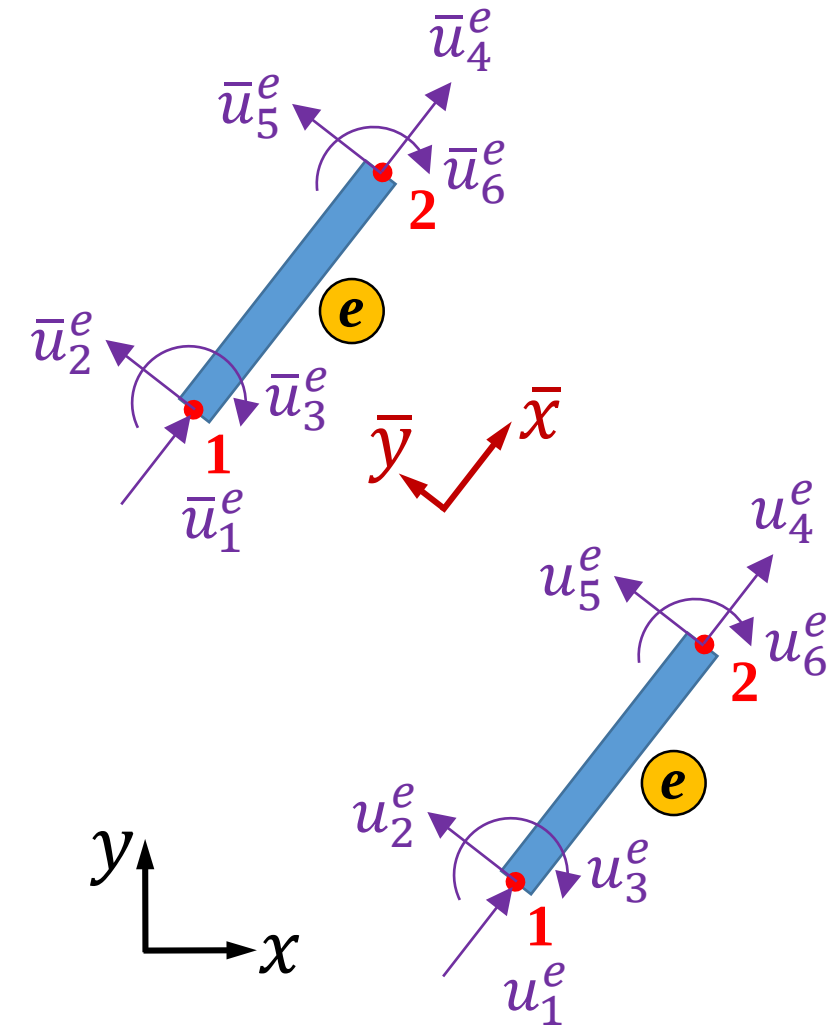
$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \\ \bar{u}_5^e \\ \bar{u}_6^e \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix}$$

$$\{\bar{u}^e\} = [T^e]\{u^e\}$$

$[T^e]$ Transformation Matrix

$$[T^e][T^e]^T = I \quad \text{Orthogonal}$$

Transformation from Local Coordinate to Global Coordinate



$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \\ \bar{u}_5^e \\ \bar{u}_6^e \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix}$$

$$\{\bar{u}^e\} = [T^e]\{u^e\}$$

Similarly

$$\{\bar{F}^e\} = [T^e]\{F^e\}$$

Transformation from Local Coordinate to Global Coordinate

$$\{\bar{u}^e\} = [T^e]\{u^e\}$$

$$[\bar{K}^e]\{\bar{u}^e\} = \{\bar{F}^e\}$$

$$\{\bar{F}^e\} = [T^e]\{F^e\}$$

$$\Rightarrow [\bar{K}^e][T^e]\{u^e\} = [T^e]\{F^e\}$$

$$[T^e][T^e]^T = I$$

$$\Rightarrow [T^e]^{-1} [\bar{K}^e][T^e]\{u^e\} = \{F^e\}$$

$$\{F^e\} = [T^e]^T\{\bar{F}^e\}$$

$$\Rightarrow [T^e]^T [\bar{K}^e][T^e]\{u^e\} = \{F^e\}$$

$$\Rightarrow [K^e]\{u^e\} = \{F^e\}$$

Summary: Frame

$$[T^e] = \begin{bmatrix} \overset{1}{\cos \alpha} & \overset{1}{\sin \alpha} & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

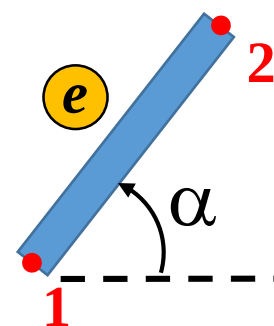
$$[K^e] = [T^e]^T [\bar{K}^e] [T^e]$$

$$\{F^e\} = [T^e]^T \{\bar{F}^e\}$$

$$[K^e]\{u^e\} = \{F^e\}$$

$$[\bar{K}^e] = \frac{2E_e I_e}{L_e^3} \begin{bmatrix} \overset{1}{\mu} & 0 & 0 & -\overset{2}{\mu} & 0 & 0 \\ 0 & 6 & -3L_e & 0 & -6 & -3L_e \\ 0 & -3L_e & 2L_e^2 & 0 & 3L_e & L_e^2 \\ \hline -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3L_e & 0 & 6 & 3L_e \\ 0 & -3L_e & L_e^2 & 0 & 3L_e & 2L_e^2 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\mu = \frac{A_e L_e^2}{2I_e}$$



Summary: Truss

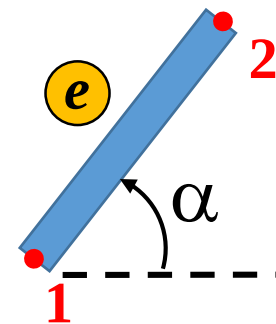
$$[T^e] = \begin{bmatrix} \overset{1}{\cos \alpha} & \overset{1}{\sin \alpha} & \overset{2}{0} & \overset{2}{0} \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ \hline 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

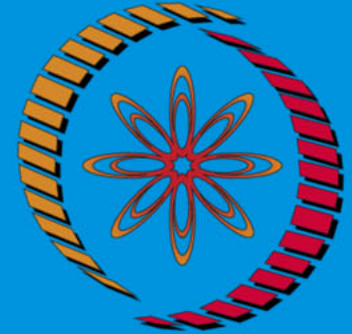
$$[K^e] = [T^e]^T [\bar{K}^e] [T^e]$$

$$\{F^e\} = [T^e]^T \{\bar{F}^e\}$$

$$[K^e]\{u^e\} = \{F^e\}$$

$$[\bar{K}^e] = \frac{A_e E_e}{L_e} \begin{bmatrix} \overset{1}{1} & \overset{1}{0} & \overset{2}{-1} & \overset{2}{0} \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$





**THANK
YOU !**



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

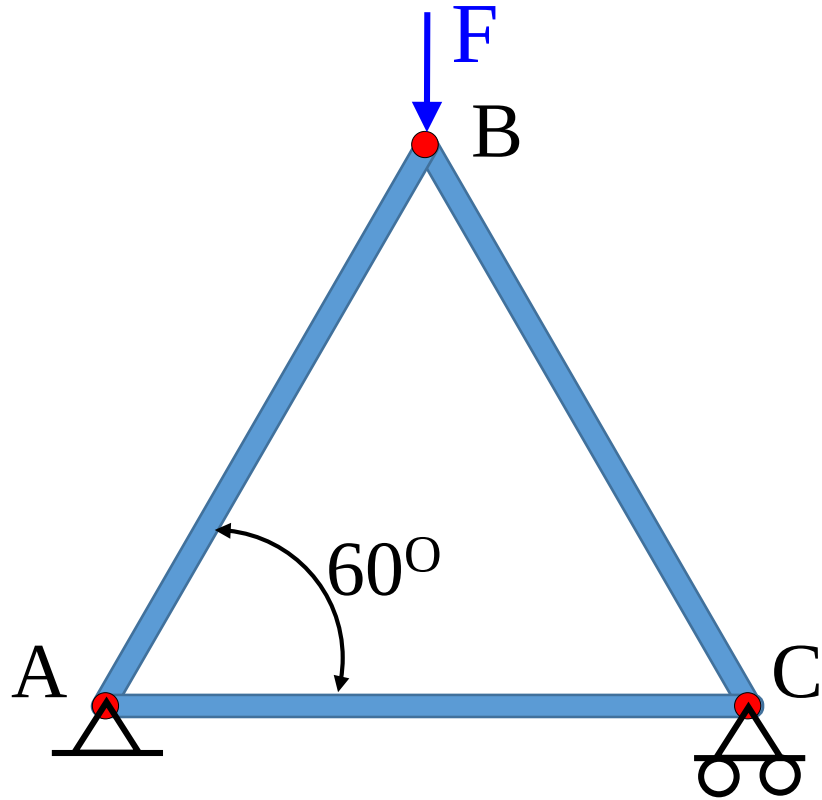
Week 6: Finite element formulation for plane trusses and frames

Lecture 28: Analysis of Plane Truss

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FE Analysis of Plane Truss: Example

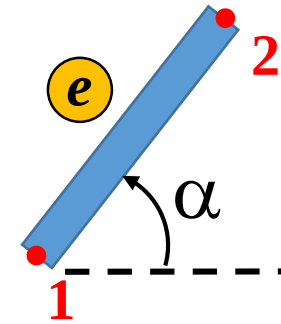


NPTEL

Recall: Truss

Element Stiffness Matrix in *Local* Coordinate

$$[\bar{K}^e] = \frac{A_e E_e}{L_e} \begin{bmatrix} \overset{1}{1} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \overset{2}{1} \\ \overset{2}{2} \\ \overset{2}{1} \\ \overset{2}{2} \end{matrix}$$



Element Stiffness Matrix in *Global* Coordinate

$$[K^e] = [T^e]^T [\bar{K}^e] [T^e]$$

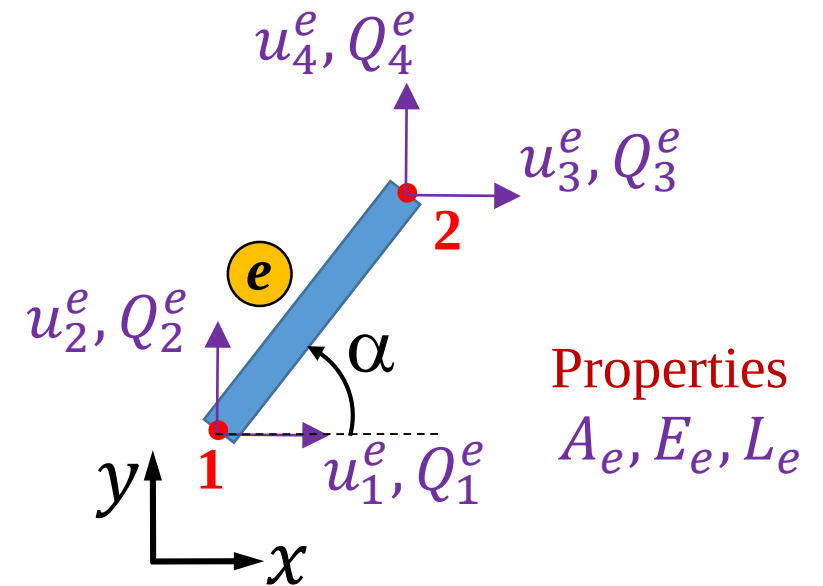
$$[T^e] = \begin{bmatrix} \overset{1}{\cos \alpha} & \overset{1}{\sin \alpha} & 0 & 0 \\ -\overset{1}{\sin \alpha} & \overset{1}{\cos \alpha} & 0 & 0 \\ 0 & 0 & \overset{2}{\cos \alpha} & \overset{2}{\sin \alpha} \\ 0 & 0 & -\overset{2}{\sin \alpha} & \overset{2}{\cos \alpha} \end{bmatrix} \begin{matrix} \overset{2}{1} \\ \overset{2}{2} \\ \overset{2}{1} \\ \overset{2}{2} \end{matrix}$$

Recall: Truss

$$[K^e] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{bmatrix}^T \frac{A_e E_e}{L_e} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\Rightarrow [K^e] = \frac{A_e E_e}{L_e} \begin{bmatrix} \lambda_x^2 & \lambda_x \lambda_y & -\lambda_x^2 & -\lambda_x \lambda_y \\ \lambda_x \lambda_y & \lambda_y^2 & -\lambda_x \lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x \lambda_y & \lambda_x^2 & \lambda_x \lambda_y \\ -\lambda_x \lambda_y & -\lambda_y^2 & \lambda_x \lambda_y & \lambda_y^2 \end{bmatrix}$$

$$\lambda_x = \cos \alpha \quad \lambda_y = \sin \alpha$$



Recall: Truss

Load Vector

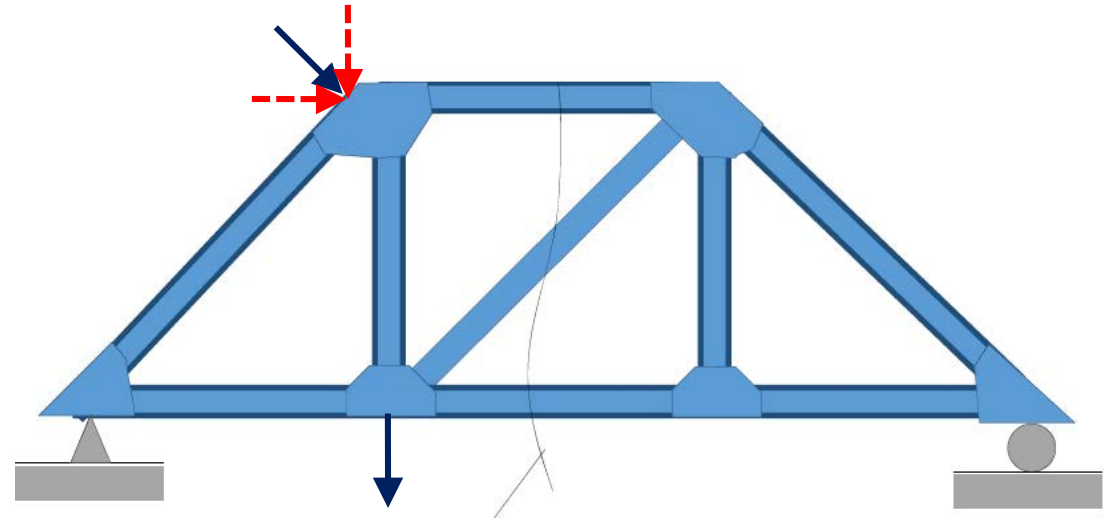
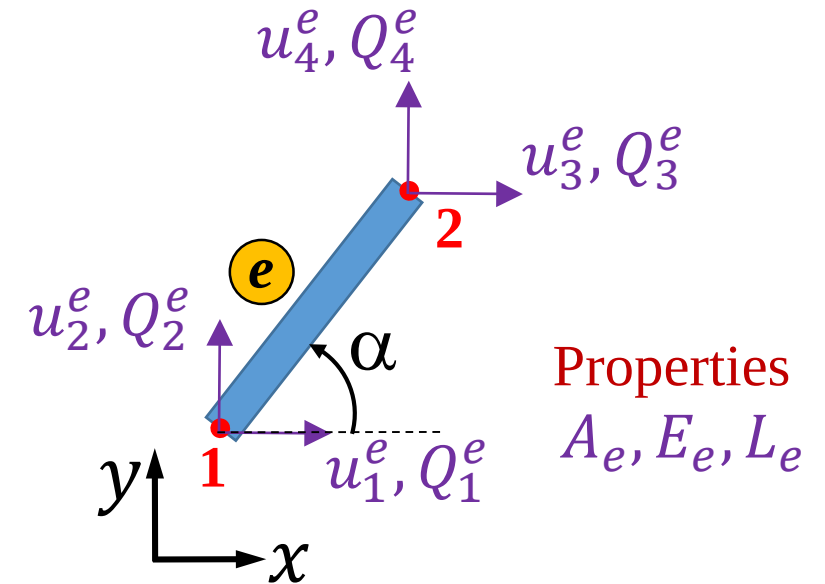
$$\{F^e\} = [T^e]^T \{\bar{F}^e\}$$

$$\Rightarrow \{Q^e\} = [T^e]^T \{\bar{Q}^e\}$$

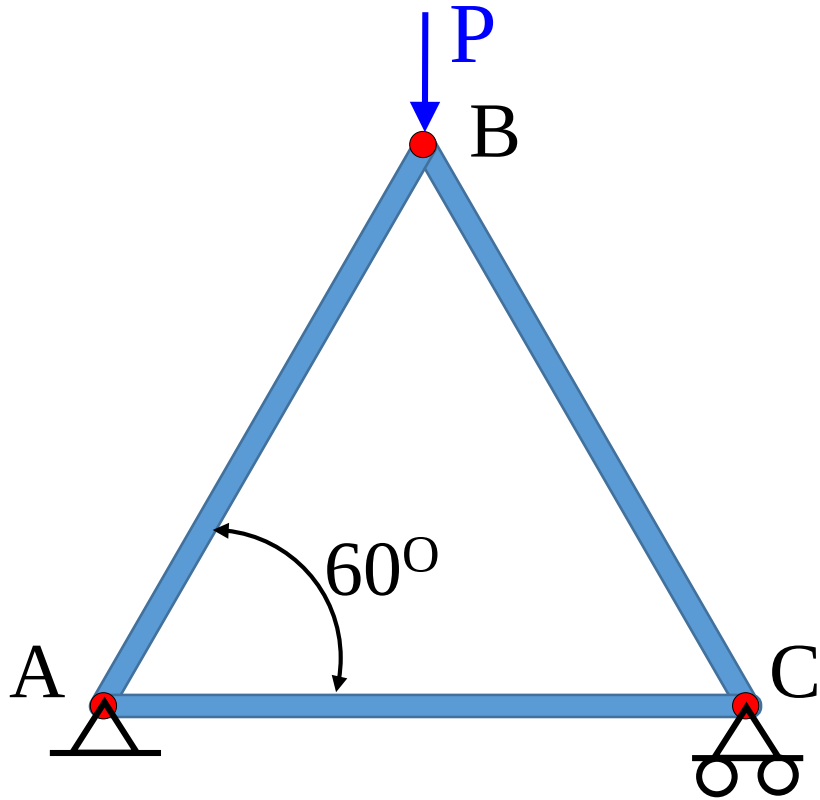
Element Equation

$$\frac{A_e E_e}{L_e} \begin{bmatrix} \lambda_x^2 & \lambda_x \lambda_y & -\lambda_x^2 & -\lambda_x \lambda_y \\ \lambda_x \lambda_y & \lambda_y^2 & -\lambda_x \lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x \lambda_y & \lambda_x^2 & \lambda_x \lambda_y \\ -\lambda_x \lambda_y & -\lambda_y^2 & \lambda_x \lambda_y & \lambda_y^2 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$\lambda_x = \sin \alpha \quad \lambda_y = \cos \alpha$$



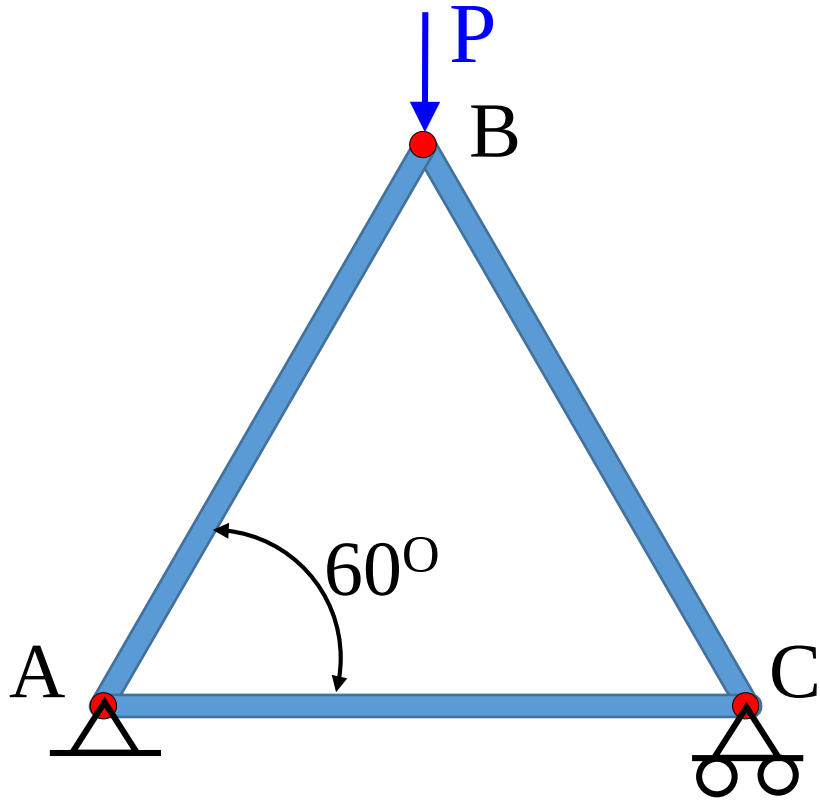
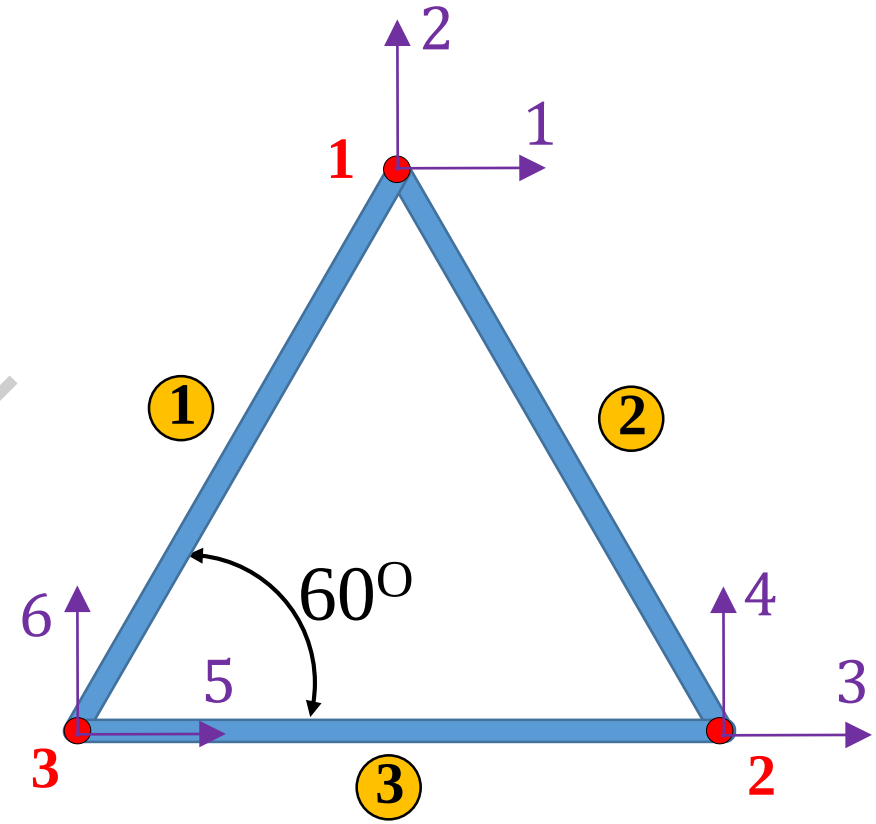
FE Analysis of Plane Truss: Example



Recall: Steps in Finite Element Method

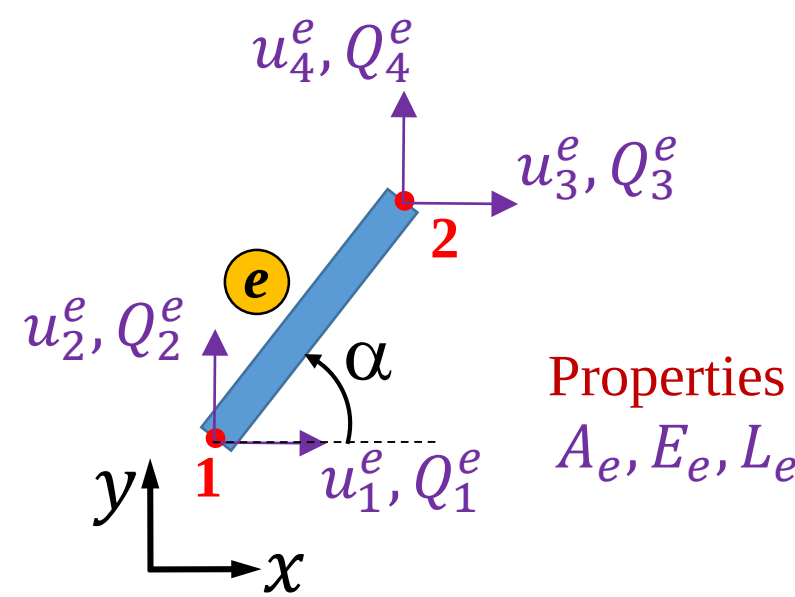
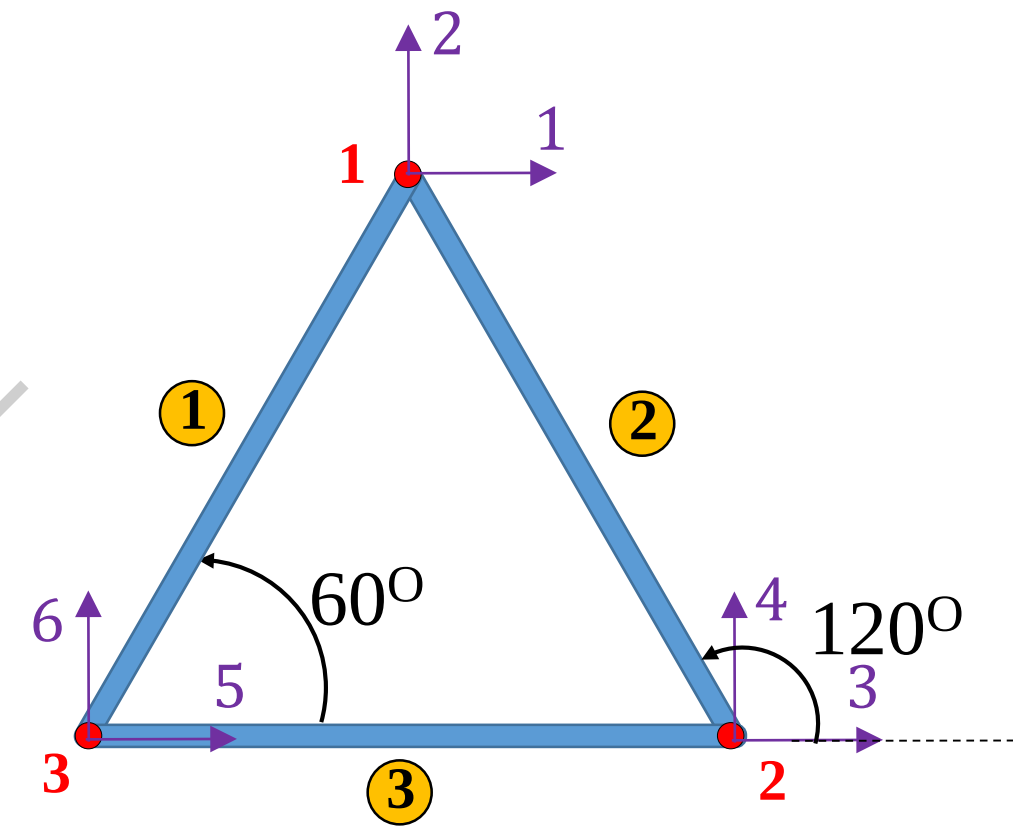
- Boundary Value Problem
 - Discretization of the Domain
 - Derivation of Element Equations
 - Assembly of Element Equations based on Element Connectivity
 - Imposition of Boundary Conditions
 - Solutions
 - Post-processing
- Today's Focus**

Step 1: Discretization



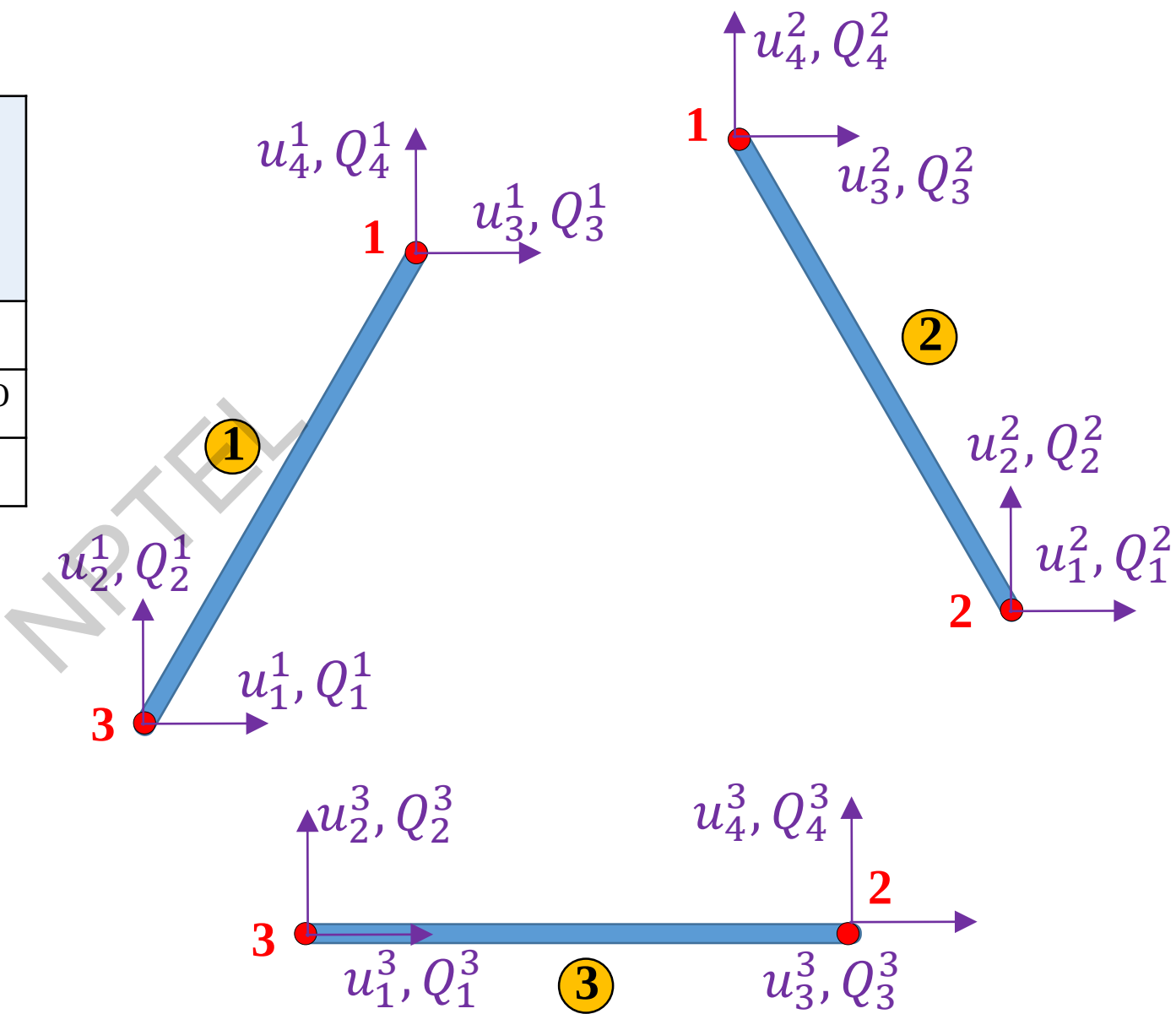
Step 1: Discretization

Element Number	Global nodes of the element (1 – 2)	Geometric and Material properties	α
1	3 – 1	A, E, L	60°
2	2 – 1	A, E, L	120°
3	3 – 2	A, E, L	0



Step 1: Discretization

Element Number	Global nodes of the element (1 – 2)	Geometric and Material properties	α
1	3 – 1	A, E, L	60°
2	2 – 1	A, E, L	120°
3	3 – 2	A, E, L	0

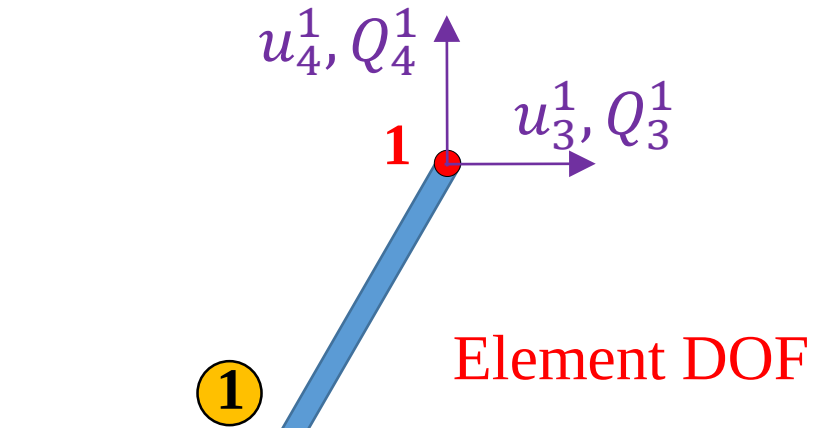
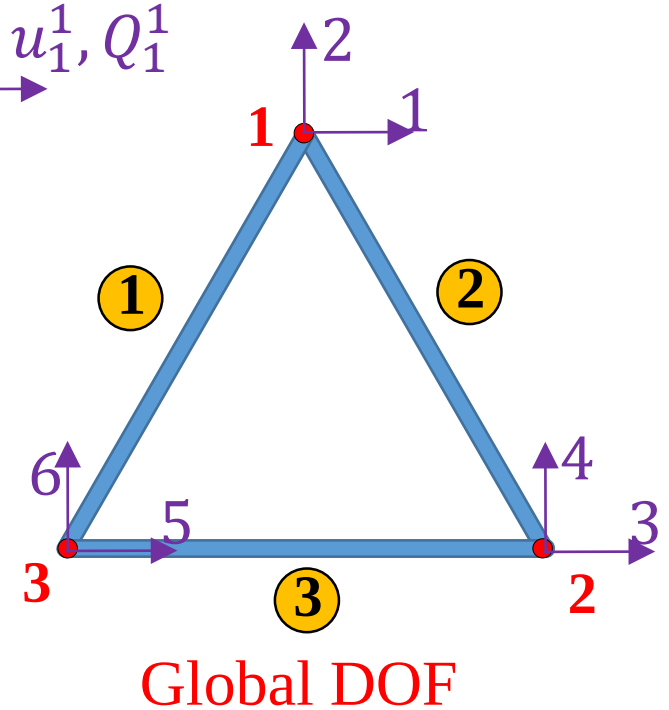


Step 2: Element Equation

Element	Nodes (1 – 2)	Properties	α	λ_x ($\cos \alpha$)	λ_y ($\sin \alpha$)
①	3 – 1	A, E, L	60°	$1/2$	$\sqrt{3}/2$

$$\frac{A_e E_e}{L_e} \begin{bmatrix} \lambda_x^2 & \lambda_x \lambda_y & -\lambda_x^2 & -\lambda_x \lambda_y \\ \lambda_x \lambda_y & \lambda_y^2 & -\lambda_x \lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x \lambda_y & \lambda_x^2 & \lambda_x \lambda_y \\ -\lambda_x \lambda_y & -\lambda_y^2 & \lambda_x \lambda_y & \lambda_y^2 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$\frac{AE}{L} \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$



Step 2: Element Equation

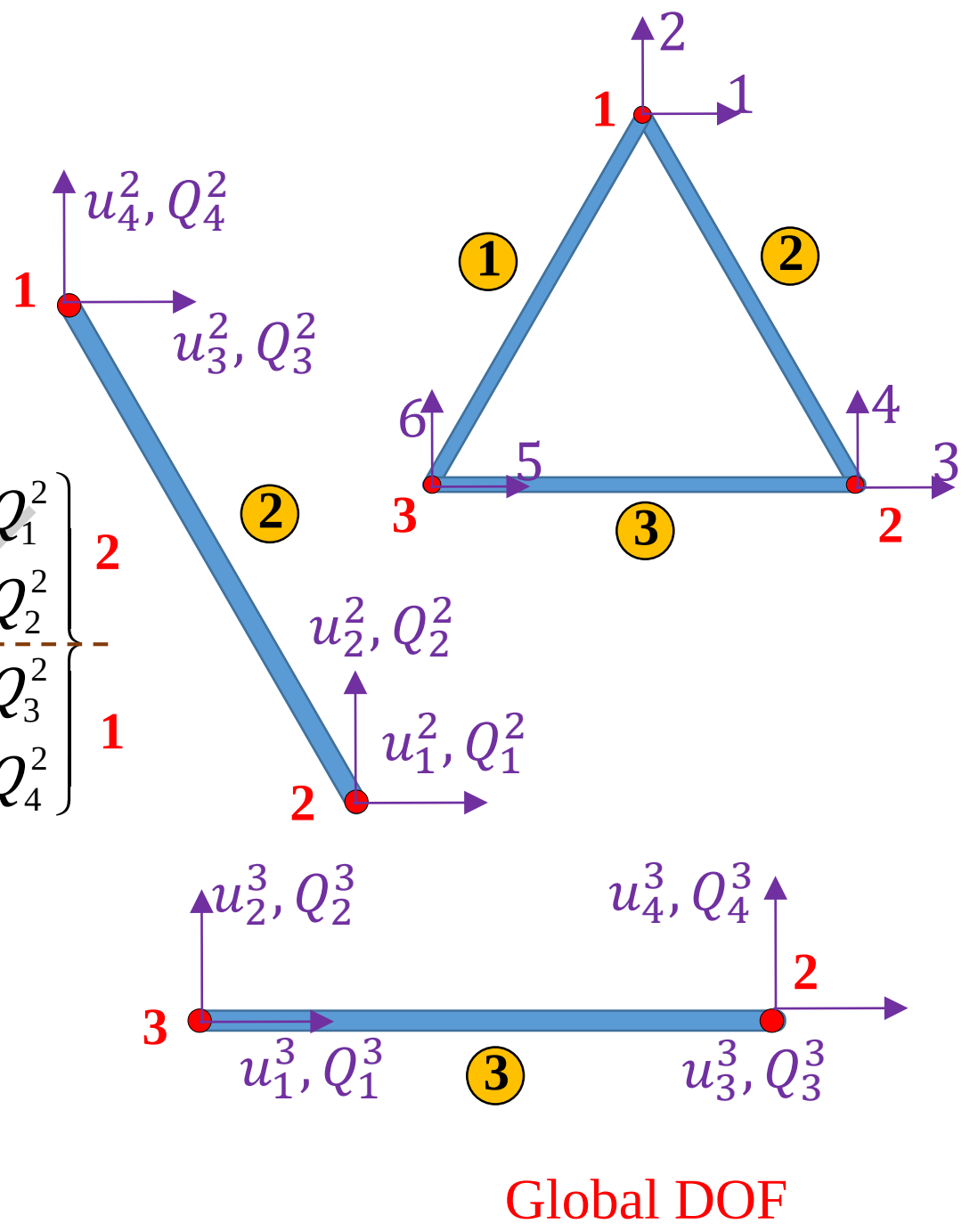
Element	Nodes (1 – 2)	Properties	α	λx ($\cos\alpha$)	λy ($\sin\alpha$)
②	2 – 1	A, E, L	120°	-1/2	$\sqrt{3}/2$
③	3 – 2	A, E, L	0°	1	0

②

$$\frac{AE}{L} \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \\ u_4^2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

③

$$\frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \\ u_4^3 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix}$$



Step 2: Element Equation

$$\textcircled{1} \frac{AE}{L} \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

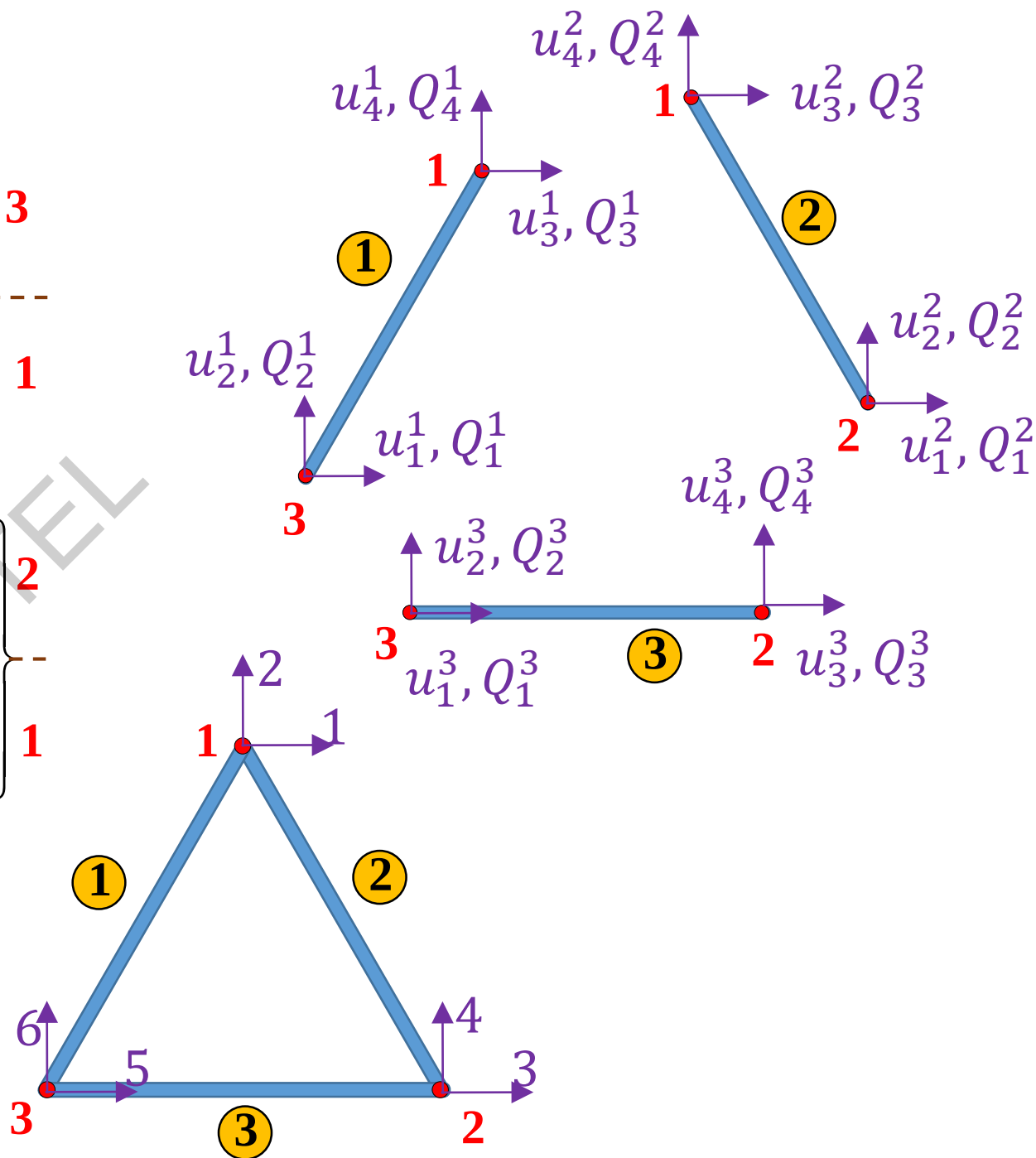
Node 1 is at the top-left, Node 2 is at the top-right, Node 3 is at the bottom-left, and Node 4 is at the bottom-right. The element is labeled 1 in a yellow circle.

$$\textcircled{2} \frac{AE}{L} \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \\ u_4^2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

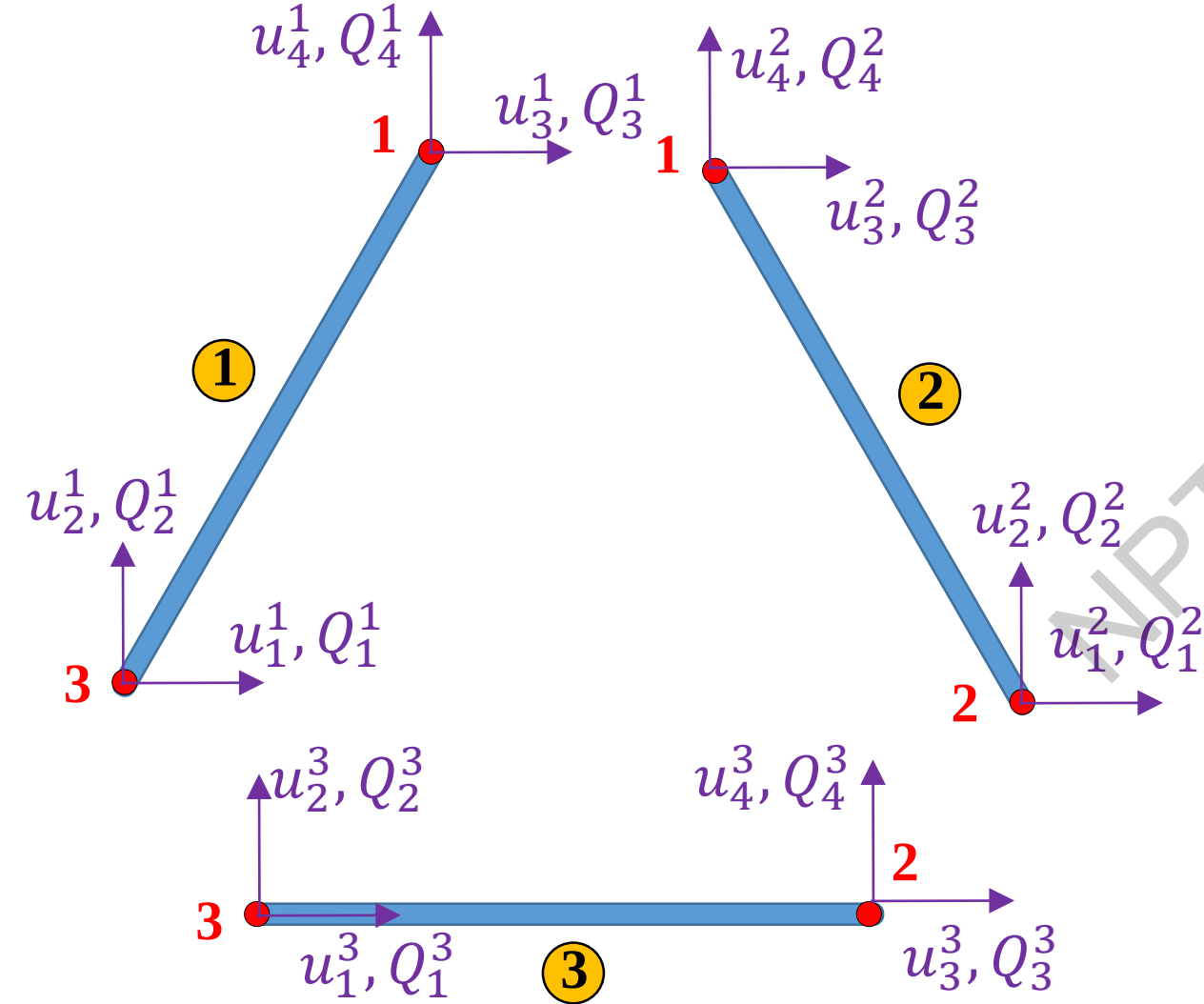
Node 1 is at the top-left, Node 2 is at the top-right, Node 3 is at the bottom-left, and Node 4 is at the bottom-right. The element is labeled 2 in a yellow circle.

$$\textcircled{3} \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \\ u_4^3 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix}$$

Node 1 is at the top-left, Node 2 is at the top-right, Node 3 is at the bottom-left, and Node 4 is at the bottom-right. The element is labeled 3 in a yellow circle.

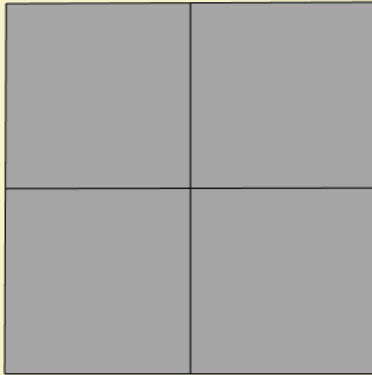


Step 3: Assembly of Element Equation Based on Element Connectivity

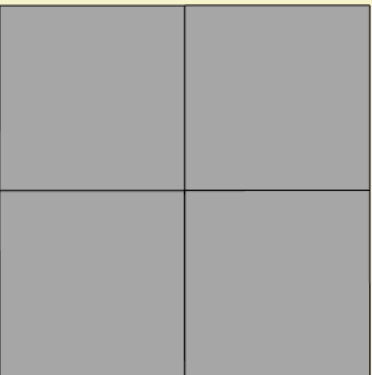


- Inter-element continuity of the primary variables (displacement compatibility)
- Inter-element equilibrium of the secondary variable (force equilibrium)

Step 3: Assembly of Element Equation Based on Element Connectivity

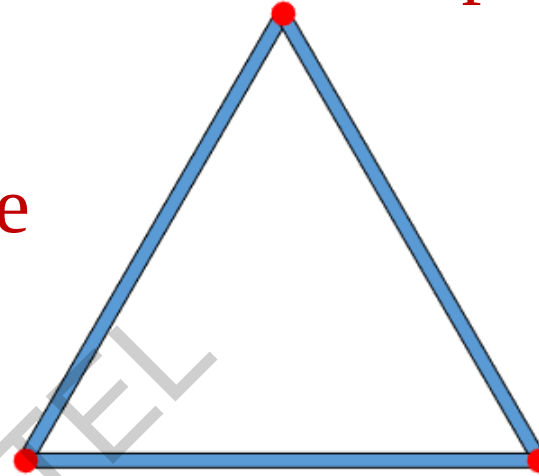
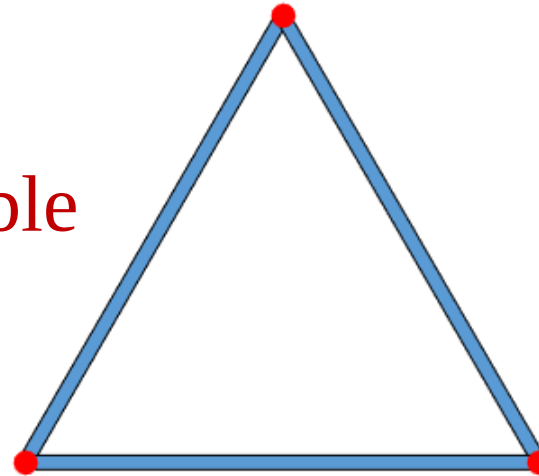


Compatible



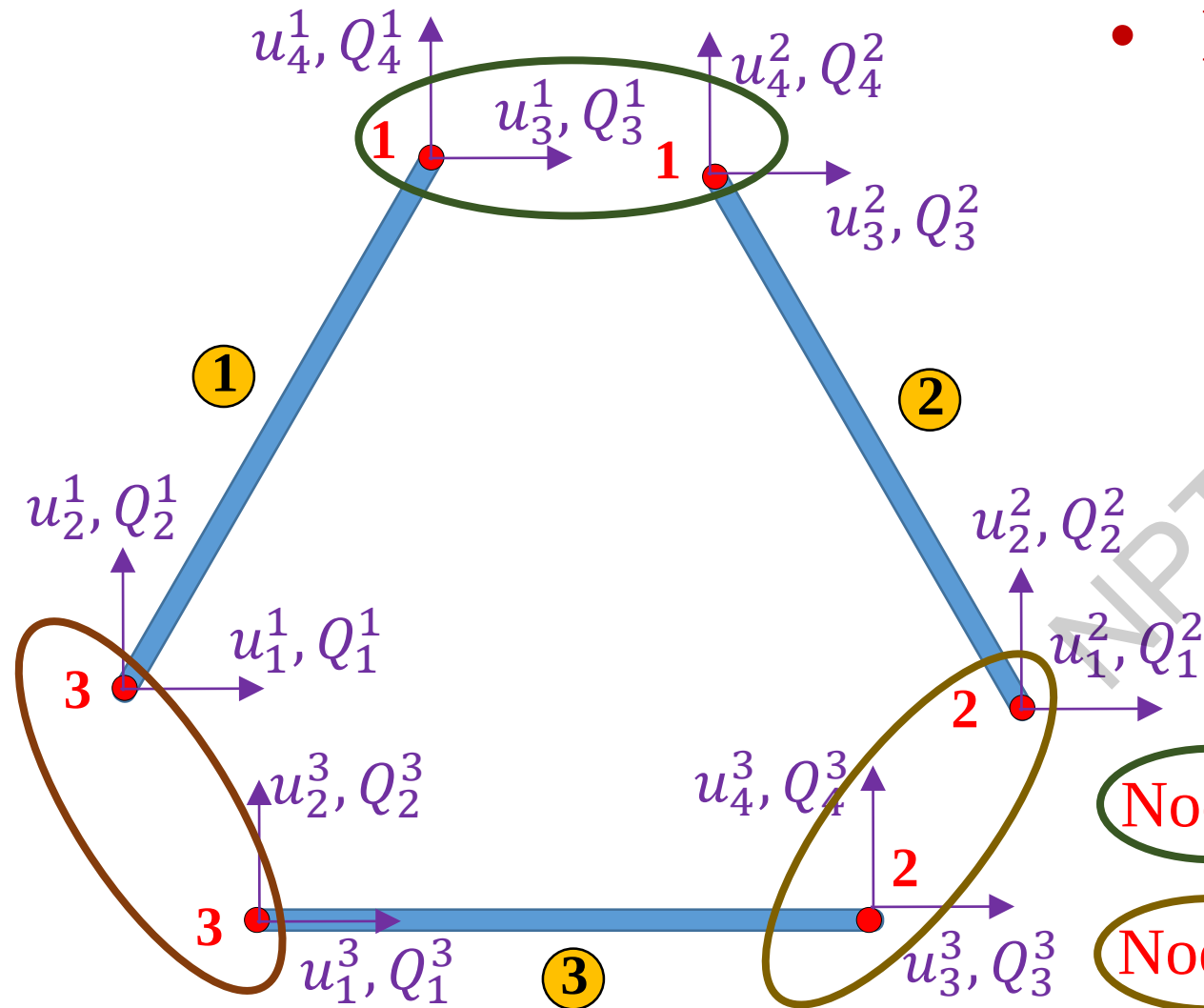
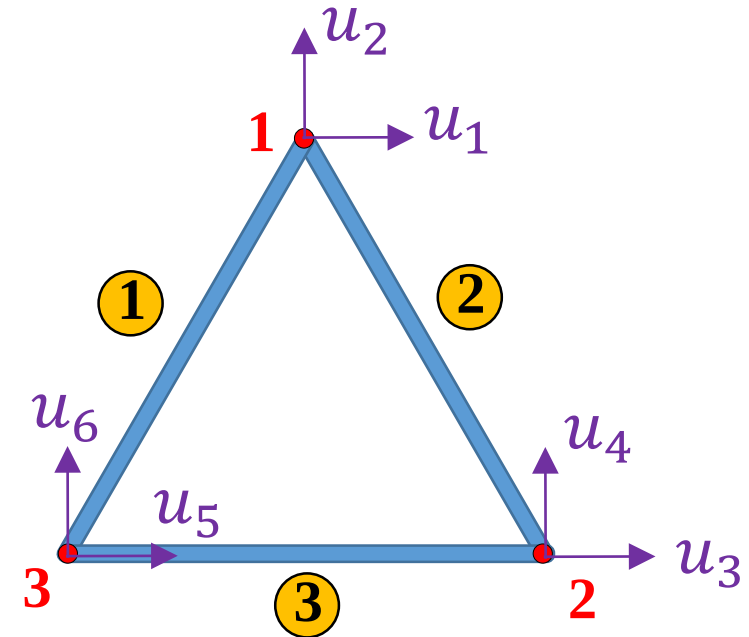
Incompatible

Displacement compatibility



Step 3: Assembly of Element Equation Based on Element Connectivity

- Displacement compatibility



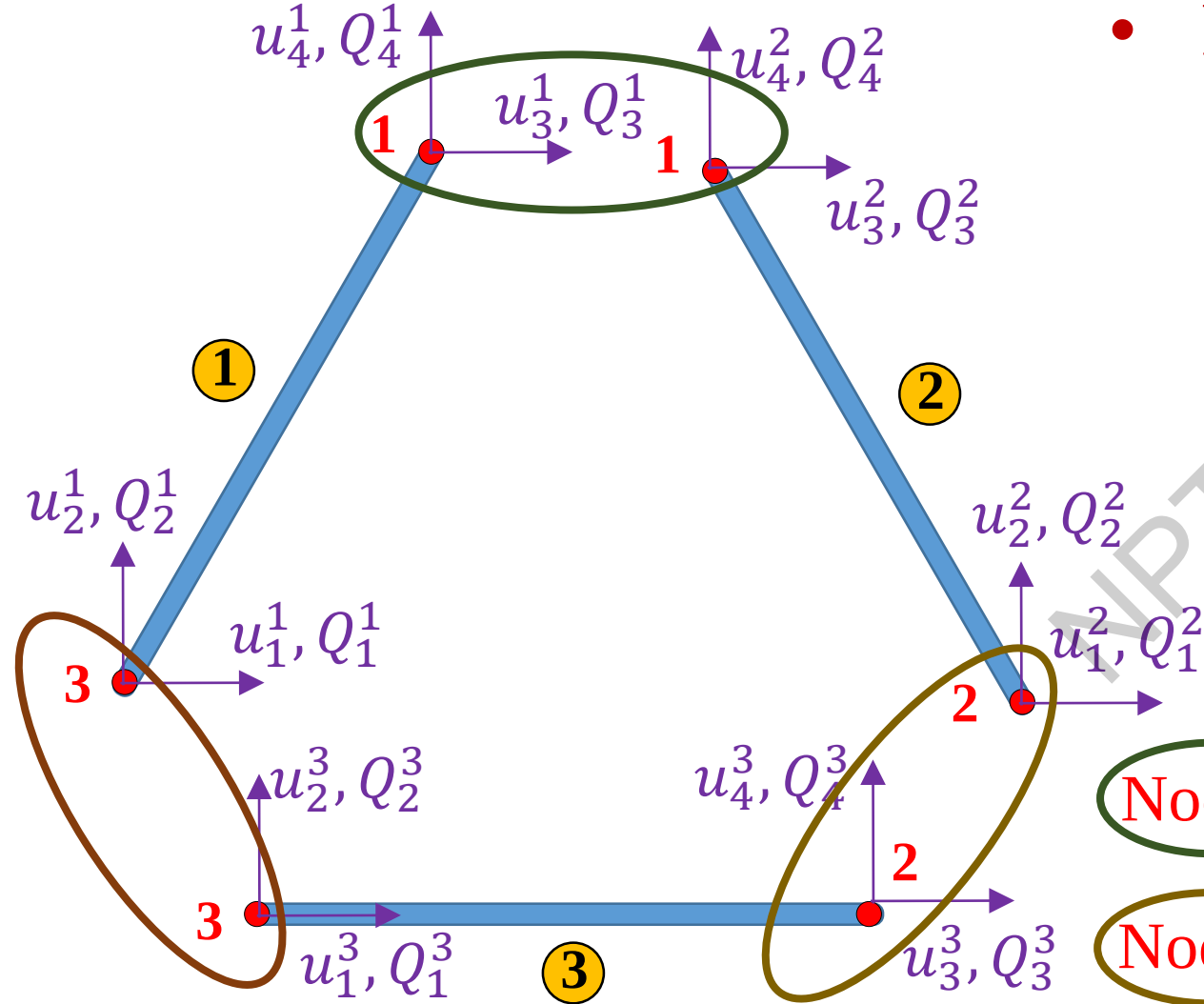
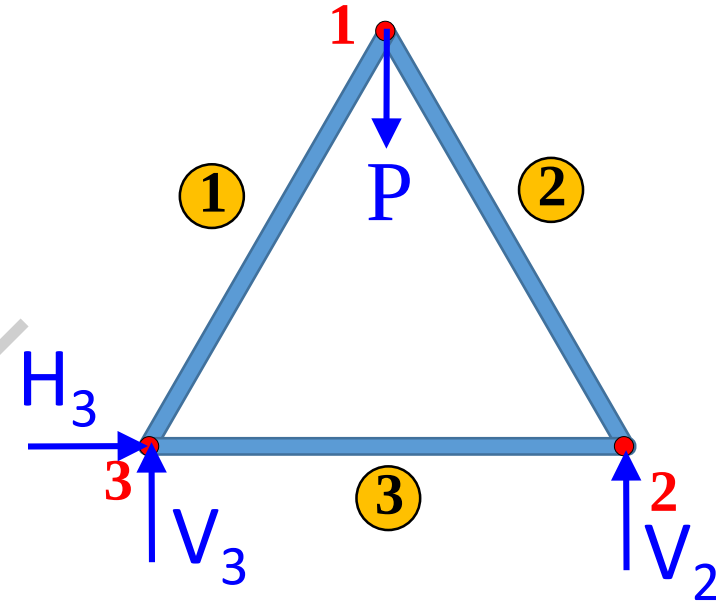
Node 1 $u_3^1 = u_3^2 = u_1$ $u_4^1 = u_4^2 = u_2$

Node 2 $u_1^2 = u_3^3 = u_3$ $u_2^2 = u_4^3 = u_4$

Node 3 $u_1^3 = u_1^1 = u_5$ $u_2^3 = u_2^1 = u_6$

Step 3: Assembly of Element Equation Based on Element Connectivity

Force Equilibrium



Node 1

$$Q_3^1 + Q_3^2 = 0 \quad Q_4^1 + Q_4^2 = -P$$

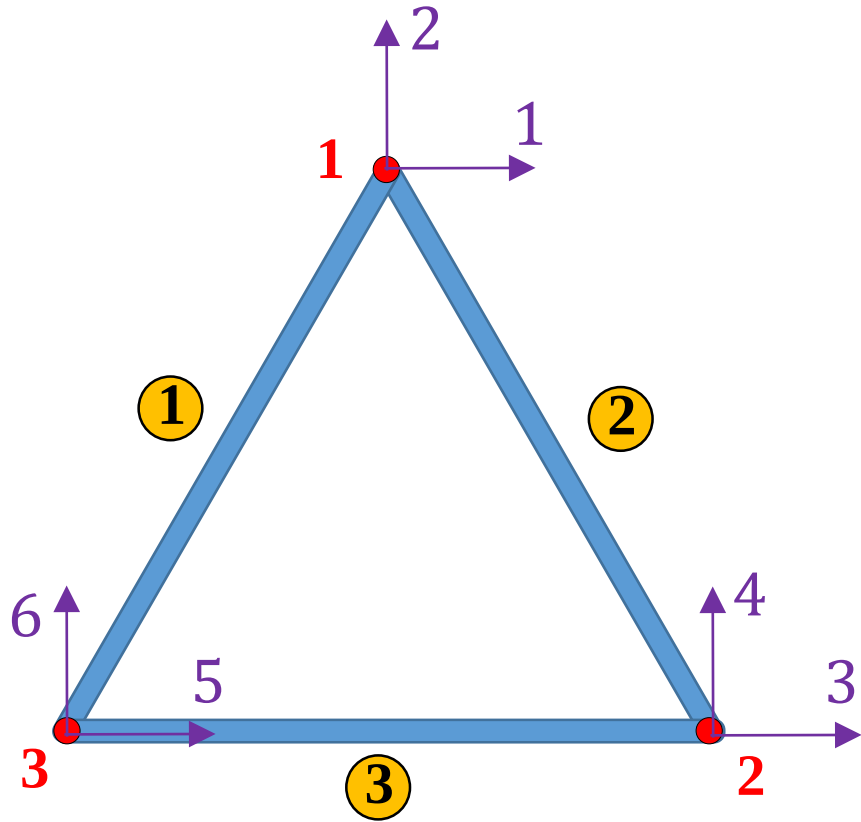
Node 2

$$Q_1^2 + Q_3^3 = 0 \quad Q_2^2 + Q_4^3 = V_2$$

Node 3

$$Q_1^3 + Q_1^1 = H_3 \quad Q_2^3 + Q_2^1 = V_3$$

Step 3: Assembly of Element Equation Based on Element Connectivity



$$[K]\{u\} = \{F\}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

Global
Displacement
Vector

$$\{F\} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \\ V_2 \\ H_3 \\ V_3 \end{Bmatrix}$$

Global Load
Vector

$$[K] \text{ Global Stiffness Matrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \left\{ \begin{array}{l} u_1^1 \rightarrow u_5 \\ u_2^1 \rightarrow u_6 \\ u_3^1 \rightarrow u_1 \\ u_4^1 \rightarrow u_2 \end{array} \right\} \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} u_1^3 \rightarrow u_5 \\ u_2^3 \rightarrow u_6 \\ u_3^3 \rightarrow u_3 \\ u_4^3 \rightarrow u_4 \end{array} \right\} \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\frac{AE}{L} \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \left\{ \begin{array}{l} u_1^2 \rightarrow u_3 \\ u_2^2 \rightarrow u_4 \\ u_3^2 \rightarrow u_1 \\ u_4^2 \rightarrow u_2 \end{array} \right\} \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

$$u_3^1 = u_3^2 = u_1$$

$$u_4^1 = u_4^2 = u_2$$

$$u_1^2 = u_3^3 = u_3$$

$$u_2^2 = u_4^3 = u_4$$

$$u_1^3 = u_1^1 = u_5$$

$$u_2^3 = u_2^1 = u_6$$

Compatibility

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\textcircled{2} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

$$u_3^1 = u_3^2 = u_1$$

$$u_4^1 = u_4^2 = u_2$$

$$u_1^2 = u_3^3 = u_3$$

$$u_2^2 = u_4^3 = u_4$$

$$u_1^3 = u_1^1 = u_5$$

$$u_2^3 = u_2^1 = u_6$$

Compatibility

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\textcircled{2} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix} \quad \text{Equilibrium}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\overset{\textcircled{1}}{\frac{AE}{L} \begin{Bmatrix} -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \end{Bmatrix}} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = Q_3^1 \quad \overset{\textcircled{2}}{\frac{AE}{L} \begin{Bmatrix} -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \end{Bmatrix}} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = Q_3^2$$

$$\frac{AE}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\overset{\textcircled{1}}{\frac{AE}{L} \begin{Bmatrix} -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \end{Bmatrix}} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = Q_3^1 \quad \overset{\textcircled{2}}{\frac{AE}{L} \begin{Bmatrix} -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \end{Bmatrix}} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = Q_3^2$$

$$\frac{AE}{L} \begin{bmatrix} 1/4 + 1/4 & \sqrt{3}/4 - \sqrt{3}/4 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\textcircled{2} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix} \quad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\overset{\textcircled{1}}{\frac{AE}{L} \begin{Bmatrix} -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{Bmatrix}} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = Q_4^1 \quad \overset{\textcircled{2}}{\frac{AE}{L} \begin{Bmatrix} \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{Bmatrix}} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = Q_4^2$$

$$\frac{AE}{L} \begin{bmatrix} 1/4 + 1/4 & \sqrt{3}/4 - \sqrt{3}/4 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 - \sqrt{3}/4 & 3/4 + 3/4 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \textcolor{red}{-1} & \textcolor{red}{0} & \textcolor{red}{1} & \textcolor{red}{0} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ \textcolor{red}{1/4} & \textcolor{red}{-\sqrt{3}/4} & \textcolor{red}{-1/4} & \textcolor{red}{\sqrt{3}/4} \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix} \quad \textcircled{2}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ \textcolor{red}{F_3} \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ \textcolor{red}{Q_1^2 + Q_3^3} \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{2} \quad \frac{AE}{L} \begin{Bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \end{Bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = Q_1^2$$

$$\textcircled{3} \quad \frac{AE}{L} \begin{Bmatrix} -1 & 0 & 1 & 0 \end{Bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = Q_3^3$$

$$\frac{AE}{L} \begin{bmatrix} 1/4+1/4 & \sqrt{3}/4-\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4-\sqrt{3}/4 & 3/4+3/4 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4+1 & -\sqrt{3}/4+0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix}$$

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\frac{AE}{L} \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} Q_3^1 + Q_3^2 \\ Q_4^1 + Q_4^2 \\ Q_1^2 + Q_3^3 \\ Q_2^2 + Q_4^3 \\ Q_1^3 + Q_1^1 \\ Q_2^3 + Q_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \\ V_2 \\ H_3 \\ V_4 \end{Bmatrix}$$

Global Stiffness Matrix

Global Displacement Vector

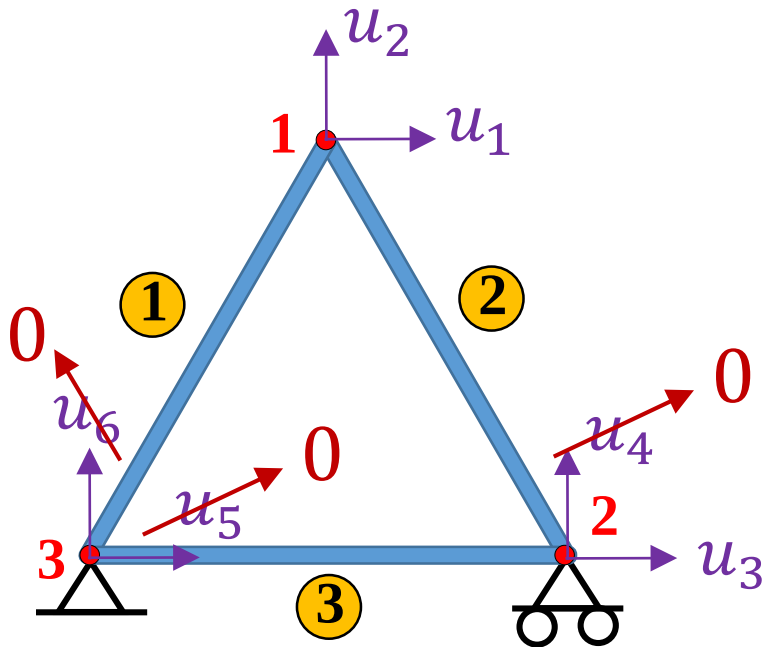
External Forces

Unknown Reactions

Step 4: Imposition of Boundary Conditions

$$\frac{AE}{L} \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \\ V_2 \\ H_3 \\ V_3 \end{Bmatrix}$$

$\xrightarrow{u_4} 0$ $\xrightarrow{u_5} 0$ $\xrightarrow{u_6} 0$



Step 4: Imposition of Boundary Conditions

$$\frac{AE}{L} \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \\ V_2 \\ H_3 \\ V_3 \end{Bmatrix}$$

Unknown Displacement

$$u_1, u_2, u_3$$

Unknown Forces

$$V_2, H_3, V_3$$

Step 5: Solution

$$\begin{array}{c} AE \\ L \end{array} \left[\begin{array}{ccc|ccc} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \hline \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{array} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \\ V_2 \\ H_3 \\ V_3 \end{Bmatrix}$$

$$\left[\begin{array}{c|c} \mathbf{K}_{ku} & \mathbf{K}_{kk} \\ \hline \mathbf{K}_{uu} & \mathbf{K}_{uk} \end{array} \right] \begin{Bmatrix} \mathbf{u}_u \\ \mathbf{u}_k \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_k \\ \mathbf{F}_u \end{Bmatrix} \Rightarrow$$

$$[\mathbf{K}_{ku}]\{\mathbf{u}_u\} + [\mathbf{K}_{kk}]\{\mathbf{u}_k\} = \{\mathbf{F}_k\}$$

Determine
Unknown
Displacement $\mathbf{u}_u$

$$[\mathbf{K}_{uu}]\{\mathbf{u}_u\} + [\mathbf{K}_{uk}]\{\mathbf{u}_k\} = \{\mathbf{F}_u\}$$

Determine
Unknown
Forces $\mathbf{F}_u$

Step 5: Solution

$$\frac{AE}{L} \begin{bmatrix} \boxed{1/2} & \boxed{0} & \boxed{-1/4} & \boxed{\sqrt{3}/4} & \boxed{-1/4} & \boxed{-\sqrt{3}/4} \\ \boxed{0} & \boxed{3/2} & \boxed{\sqrt{3}/4} & \boxed{-3/4} & \boxed{-\sqrt{3}/4} & \boxed{-3/4} \\ \boxed{-1/4} & \boxed{\sqrt{3}/4} & \boxed{5/4} & \boxed{-\sqrt{3}/4} & \boxed{-1} & \boxed{0} \\ \hline \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{bmatrix} \boxed{u_1} \\ \boxed{u_2} \\ \boxed{u_3} \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \boxed{0} \\ \boxed{-P} \\ \boxed{0} \\ \hline V_2 \\ H_3 \\ V_3 \end{bmatrix}$$

$$\frac{AE}{L} \begin{bmatrix} 1/2 & 0 & -1/4 \\ 0 & 3/2 & \sqrt{3}/4 \\ -1/4 & \sqrt{3}/4 & 5/4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{AE} \begin{Bmatrix} 0.1433 \\ -0.7500 \\ 0.2887 \end{Bmatrix}$$

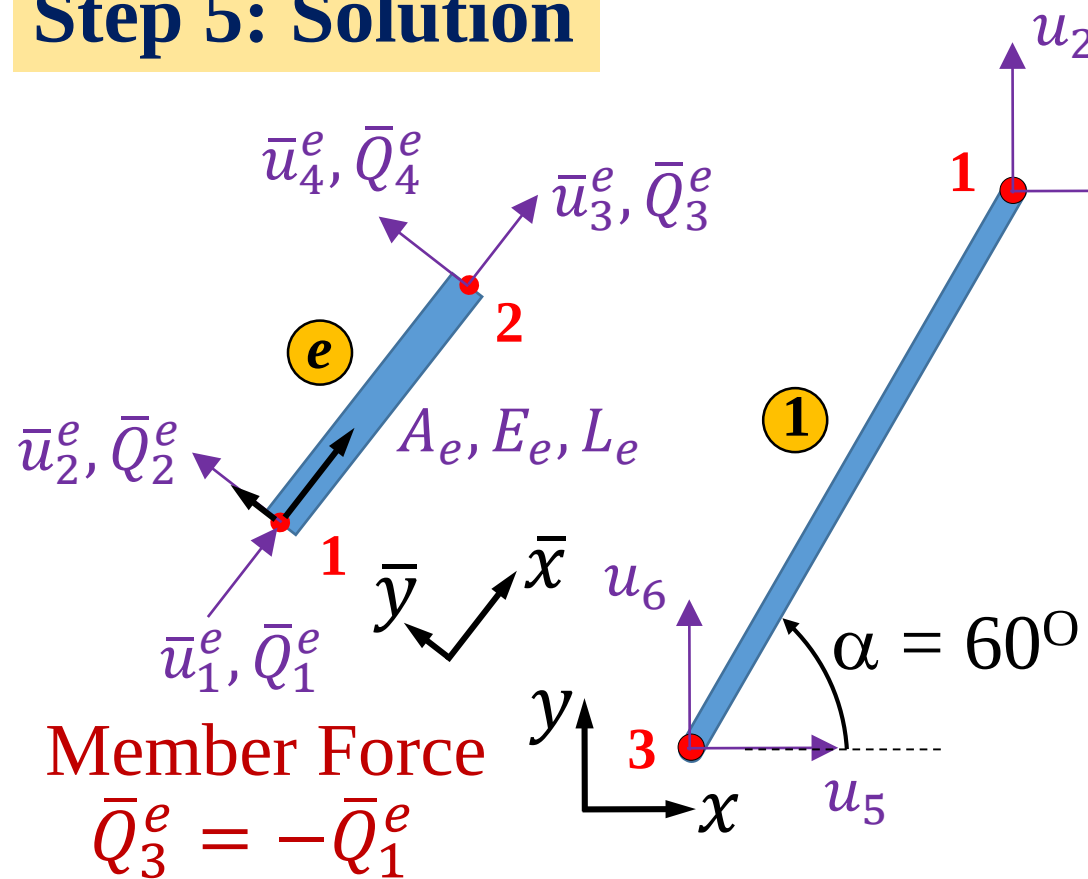
Step 5: Solution

$$\frac{AE}{L} \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \hline \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \\ V_2 \\ H_3 \\ V_3 \end{Bmatrix}$$

$$\frac{AE}{L} \begin{bmatrix} \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 \\ -1/4 & -\sqrt{3}/4 & -1 \\ -\sqrt{3}/4 & -3/4 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} V_2 \\ H_3 \\ V_3 \end{Bmatrix}$$

$$\Rightarrow \frac{AE}{L} \begin{bmatrix} \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 \\ -1/4 & -\sqrt{3}/4 & -1 \\ -\sqrt{3}/4 & -3/4 & 0 \end{bmatrix} \frac{PL}{AE} \begin{Bmatrix} 0.1433 \\ -0.7500 \\ 0.2887 \end{Bmatrix} = \begin{Bmatrix} V_2 \\ H_3 \\ V_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} V_2 \\ H_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0.5P \\ 0 \\ 0.5P \end{Bmatrix}$$

Step 5: Solution



$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix}$$

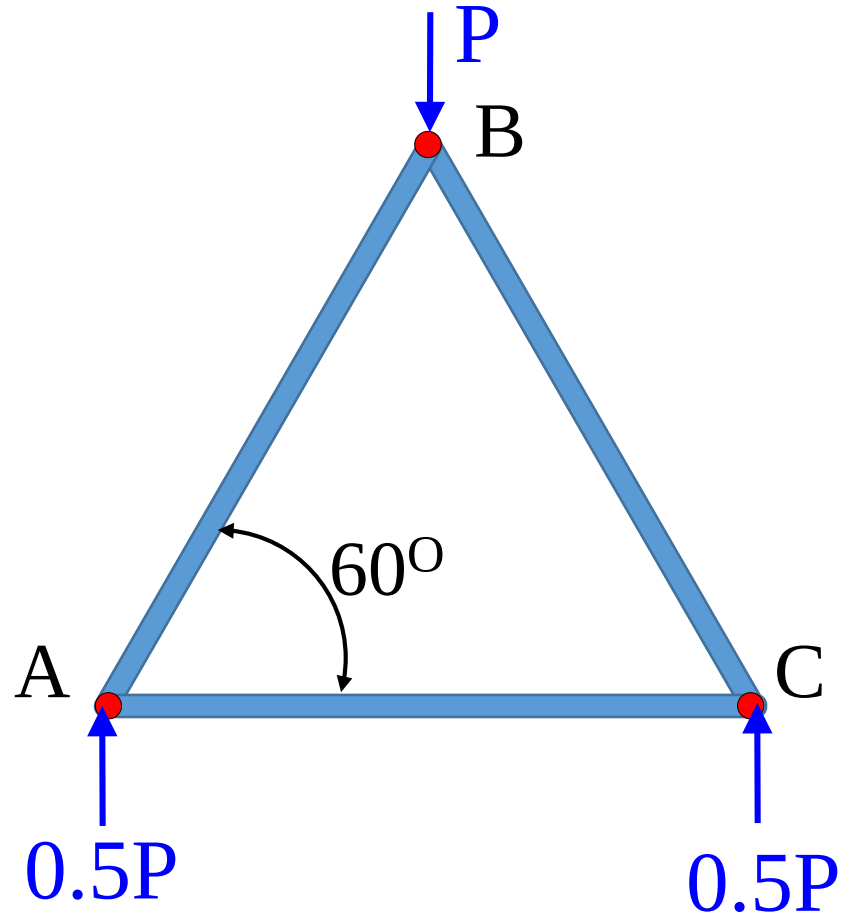
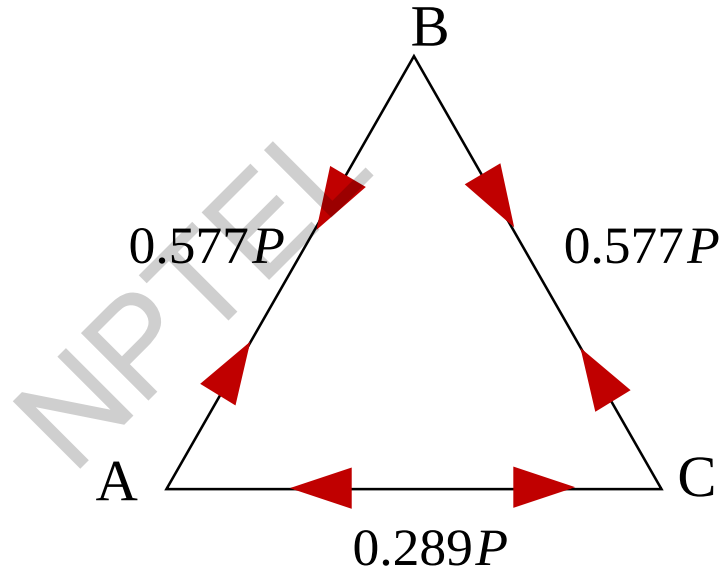
$$\begin{aligned} \bar{Q}_3^e &= \frac{AE}{L} (-\bar{u}_1^e + \bar{u}_3^e) \\ \Rightarrow \bar{Q}_3^e &= \frac{AE}{L} \{-\cos \alpha \quad -\sin \alpha \quad \cos \alpha \quad \sin \alpha\} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} \end{aligned}$$

$$\frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \end{Bmatrix} = \begin{Bmatrix} \bar{Q}_1^e \\ \bar{Q}_2^e \\ \bar{Q}_3^e \\ \bar{Q}_4^e \end{Bmatrix}$$

$$\Rightarrow \bar{Q}_3^e = \frac{AE}{L} \left\{ -\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2} \right\} \frac{PL}{AE} \begin{Bmatrix} 0 \\ 0 \\ 0.1433 \\ -0.7500 \end{Bmatrix}$$

$$\Rightarrow \bar{Q}_3^e = -0.577P$$

Step 5: Solution



Element Force Diagram



**THANK
YOU !**



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

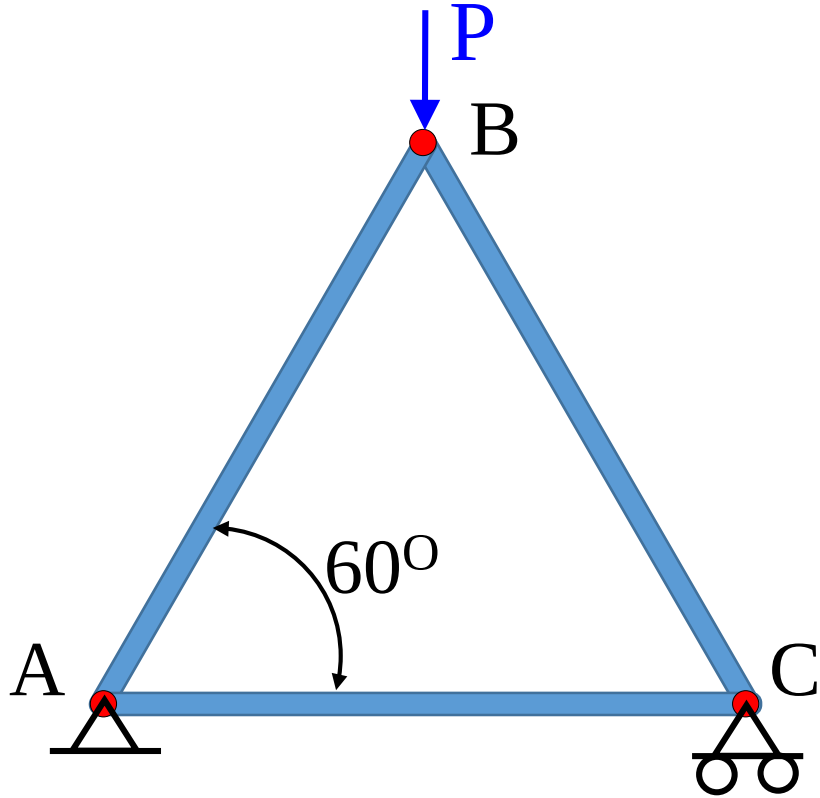
Week 6: Finite element formulation for plane trusses and frames

Lecture 29: Analysis of Plane Truss (Computer Implementation)

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Amit Shaw
Associate Professor
Department of Civil Engineering
IIT Kharagpur

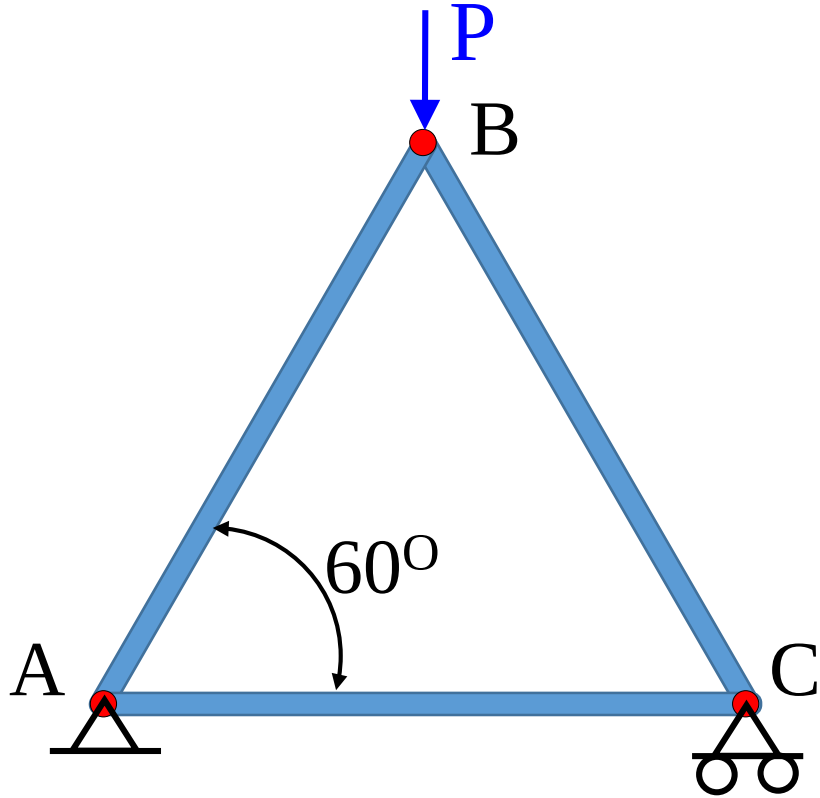
FE Analysis of Plane Truss: Example



Recall: Steps in Finite Element Method

- Boundary Value Problem
- Discretization of the Domain
- Derivation of Element Equations
- Assembly of Element Equations based on Element Connectivity
- Imposition of Boundary Conditions
- Solutions
- Post-processing

FE Analysis of Plane Truss: Example



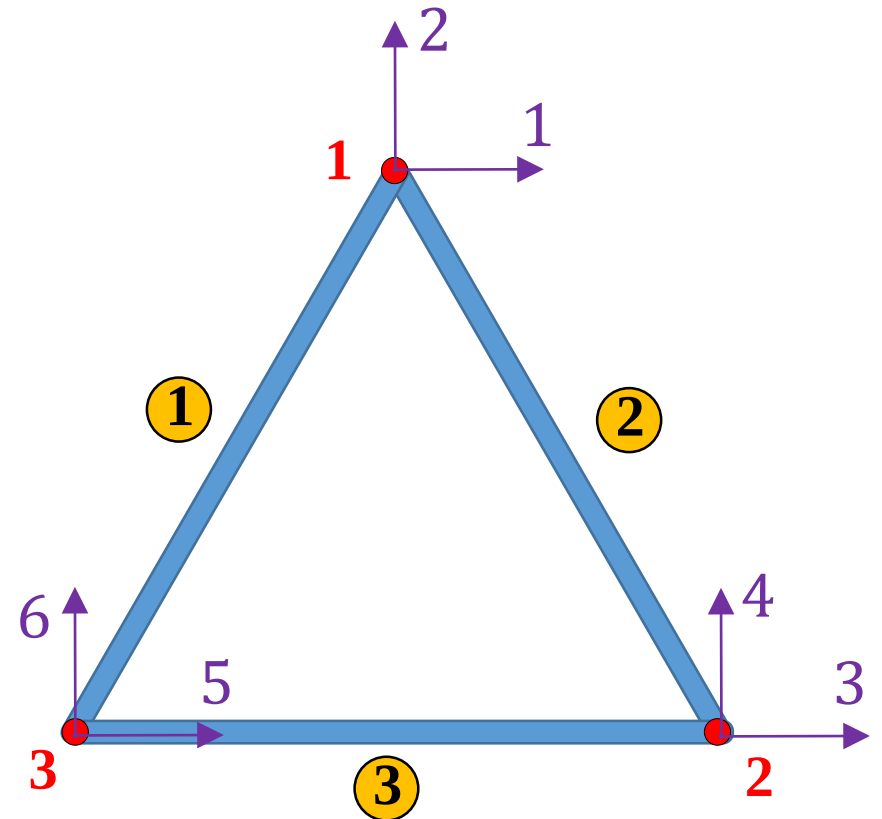
Recall: Steps in Finite Element Method

- Boundary Value Problem
- Discretization of the Domain
- Derivation of Element Equations
- Assembly of Element Equations based on Element Connectivity
- Imposition of Boundary Conditions
- Solutions
- Post-processing

Step 3: Assembly of Element Equation Based on Element Connectivity

$$\textcircled{1} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix} \quad \frac{AE}{L} \begin{bmatrix} \textcolor{blue}{5} & \textcolor{blue}{6} & \textcolor{blue}{3} & \textcolor{blue}{4} \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix} \quad \textcircled{3}$$

$$\frac{AE}{L} \begin{bmatrix} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$



$$\begin{bmatrix} K^1 \end{bmatrix} = \begin{matrix} & \begin{matrix} \text{5} & \text{6} & \text{1} & \text{2} \end{matrix} \\ \begin{matrix} \frac{AE}{L} \end{matrix} & \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix} \begin{matrix} \text{5} \\ \text{6} \\ \text{1} \\ \text{2} \end{matrix}$$

$$\left[\begin{array}{c} K^2 \\ \frac{AE}{L} \end{array} \right] = \left[\begin{array}{cccc} \textcolor{blue}{3} & \textcolor{blue}{4} & \textcolor{blue}{1} & \textcolor{blue}{2} \\ 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{array} \right]$$

$$\begin{bmatrix} K^3 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is a 6x6 grid with the following structure:

- Diagonal elements: 2
- Off-diagonal elements: 1 for adjacent nodes (1-2, 2-1, 2-3, 3-2, 3-4, 4-3, 4-5, 5-4, 5-6, 6-5, 6-1, 1-6)
- Block structure: The matrix is partitioned into four 3x3 blocks by dashed red lines.

The matrix is labeled $[K] = \frac{AE}{L}$ on the left.

$$[K^1] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 1 & 2 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & \boxed{1/4} & \boxed{\sqrt{3}/4} \\ -\sqrt{3}/4 & -3/4 & \boxed{\sqrt{3}/4} & \boxed{3/4} \end{bmatrix} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & \boxed{1/4} & \boxed{-\sqrt{3}/4} \\ \sqrt{3}/4 & -3/4 & \boxed{-\sqrt{3}/4} & \boxed{3/4} \end{bmatrix} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 3 & 4 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \begin{matrix} 1 & 2 \\ \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{matrix} & \begin{matrix} 3 & 4 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 5 & 6 \\ 0 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}$$

$$[K^1] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{matrix} \overset{5}{5} \\ \overset{6}{6} \\ \overset{1}{1} \\ \overset{2}{2} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{bmatrix} \overset{3}{1/4} & \overset{4}{-\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{\sqrt{3}/4} \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ \boxed{-1/4} & \boxed{\sqrt{3}/4} & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{matrix} \overset{3}{3} \\ \overset{4}{4} \\ \overset{1}{1} \\ \overset{2}{2} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1} & \overset{6}{0} & \overset{3}{-1} & \overset{4}{0} \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \overset{5}{5} \\ \overset{6}{6} \\ \overset{3}{3} \\ \overset{4}{4} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \overset{1}{1/2} & \overset{2}{0} & \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{0} & \overset{6}{0} \\ 0 & \overset{2}{3/2} & \overset{3}{\sqrt{3}/4} & \overset{4}{-3/4} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \overset{1}{1} \\ \overset{2}{2} \\ \overset{3}{3} \\ \overset{4}{4} \\ \overset{5}{5} \\ \overset{6}{6} \end{matrix}$$

$$[K^1] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} & \overset{5}{0} \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 & \overset{6}{0} \\ \boxed{-1/4} & \boxed{-\sqrt{3}/4} & 1/4 & \sqrt{3}/4 & \overset{1}{0} \\ \boxed{-\sqrt{3}/4} & \boxed{-3/4} & \sqrt{3}/4 & 3/4 & \overset{2}{0} \end{bmatrix}$$

$$[K^2] = \frac{AE}{L} \begin{bmatrix} \overset{3}{1/4} & \overset{4}{-\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{\sqrt{3}/4} & \overset{3}{0} \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 & \overset{4}{0} \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 & \overset{1}{0} \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & \overset{2}{0} \end{bmatrix}$$

$$[K^3] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1} & \overset{6}{0} & \overset{3}{-1} & \overset{4}{0} & \overset{5}{0} \\ 0 & 0 & 0 & 0 & \overset{6}{0} \\ -1 & 0 & 1 & 0 & \overset{3}{0} \\ 0 & 0 & 0 & 0 & \overset{4}{0} \end{bmatrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \overset{1}{1/2} & \overset{2}{0} & \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{-1/4} & \overset{6}{-\sqrt{3}/4} & \overset{1}{0} \\ 0 & \overset{2}{3/2} & \overset{3}{\sqrt{3}/4} & \overset{4}{-3/4} & \overset{5}{-\sqrt{3}/4} & \overset{6}{-3/4} & \overset{2}{0} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \overset{3}{0} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \overset{4}{0} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \overset{5}{0} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \overset{6}{0} \end{bmatrix}$$

$$[K^1] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \begin{matrix} \overset{5}{5} \\ \overset{6}{6} \\ \overset{1}{1} \\ \overset{2}{2} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{bmatrix} \overset{3}{1/4} & \overset{4}{-\sqrt{3}/4} & \boxed{\overset{1}{-1/4} \quad \overset{2}{\sqrt{3}/4}} \\ -\sqrt{3}/4 & 3/4 & \boxed{\sqrt{3}/4 \quad -3/4} \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \begin{matrix} \overset{3}{3} \\ \overset{4}{4} \\ \overset{1}{1} \\ \overset{2}{2} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1} & \overset{6}{0} & \overset{3}{-1} & \overset{4}{0} \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \overset{5}{5} \\ \overset{6}{6} \\ \overset{3}{3} \\ \overset{4}{4} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \overset{1}{1/2} & \overset{2}{0} & \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{-1/4} & \overset{6}{-\sqrt{3}/4} \\ 0 & \frac{3}{2} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \overset{1}{1} \\ \overset{2}{2} \\ \overset{3}{3} \\ \overset{4}{4} \\ \overset{5}{5} \\ \overset{6}{6} \end{matrix}$$

$$[K^1] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 1 & 2 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 3 & 4 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \begin{matrix} 1 & 2 \\ 1/2 & 0 \\ 0 & 3/2 \end{matrix} & \begin{matrix} 3 & 4 \\ \sqrt{3}/4 & -3/4 \\ \sqrt{3}/4 & -3/4 \end{matrix} & \begin{matrix} 5 & 6 \\ -1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 \end{matrix} \\ \begin{matrix} 3 & 4 \\ -1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 \end{matrix} & \begin{matrix} 1 & 2 \\ 5/4 & -\sqrt{3}/4 \\ -3/4 & 3/4 \end{matrix} & \begin{matrix} 5 & 6 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 5 & 6 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 3 & 4 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 2 \\ -1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 \end{matrix} \end{bmatrix}$$

$$[K^1] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} \\ \overset{5}{\sqrt{3}/4} & \overset{6}{3/4} & \overset{1}{-\sqrt{3}/4} & \overset{2}{-3/4} \\ \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} & \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} \\ \overset{1}{-\sqrt{3}/4} & \overset{2}{-3/4} & \overset{5}{\sqrt{3}/4} & \overset{6}{3/4} \end{bmatrix}$$

$$[K^2] = \frac{AE}{L} \begin{bmatrix} \overset{3}{1/4} & \overset{4}{-\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{\sqrt{3}/4} \\ \overset{3}{-\sqrt{3}/4} & \overset{4}{3/4} & \overset{1}{\sqrt{3}/4} & \overset{2}{-3/4} \\ \overset{1}{-1/4} & \overset{2}{\sqrt{3}/4} & \overset{3}{1/4} & \overset{4}{-\sqrt{3}/4} \\ \overset{1}{\sqrt{3}/4} & \overset{2}{-3/4} & \overset{3}{-\sqrt{3}/4} & \overset{4}{3/4} \end{bmatrix}$$

$$[K^3] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1} & \overset{6}{0} & \overset{3}{-1} & \overset{4}{0} \\ \overset{5}{0} & \overset{6}{0} & \overset{3}{0} & \overset{4}{0} \\ \overset{3}{-1} & \overset{4}{0} & \overset{5}{1} & \overset{6}{0} \\ \overset{3}{0} & \overset{4}{0} & \overset{5}{0} & \overset{6}{0} \end{bmatrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \overset{1}{1/2} & \overset{2}{0} & \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{-1/4} & \overset{6}{-\sqrt{3}/4} \\ 0 & \overset{2}{3/2} & \overset{3}{\sqrt{3}/4} & \overset{4}{-3/4} & \overset{5}{-\sqrt{3}/4} & \overset{6}{-3/4} \\ -\overset{3}{1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{5/4} & \overset{6}{-\sqrt{3}/4} & -1 & 0 \\ \overset{4}{\sqrt{3}/4} & \overset{2}{-3/4} & \overset{3}{-\sqrt{3}/4} & \overset{4}{3/4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K^1] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 1 & 2 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 3 & 4 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & 0 & 0 & 0 & 0 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K^1] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 1 & 2 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 3 & 4 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 0 & 0 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K^1] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 1 & 2 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^2] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^3] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 5 & 6 & 3 & 4 \end{matrix} \\ \begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$[K] = \frac{AE}{L} \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1/2 & 0 & -1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ 0 & 3/2 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 5/4 & -\sqrt{3}/4 & -1 & 0 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & -1 & 0 & 5/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \end{matrix}$$

$$[K^1] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} \\ \overset{5}{\sqrt{3}/4} & \overset{6}{3/4} & \overset{1}{-\sqrt{3}/4} & \overset{2}{-3/4} \\ \overset{1}{-1/4} & \overset{2}{-\sqrt{3}/4} & \overset{5}{1/4} & \overset{6}{\sqrt{3}/4} \\ \overset{1}{-\sqrt{3}/4} & \overset{2}{-3/4} & \overset{5}{\sqrt{3}/4} & \overset{6}{3/4} \end{bmatrix}$$

$$[K^2] = \frac{AE}{L} \begin{bmatrix} \overset{3}{1/4} & \overset{4}{-\sqrt{3}/4} & \overset{1}{-1/4} & \overset{2}{\sqrt{3}/4} \\ \overset{3}{-\sqrt{3}/4} & \overset{4}{3/4} & \overset{1}{\sqrt{3}/4} & \overset{2}{-3/4} \\ \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{1}{1/4} & \overset{2}{-\sqrt{3}/4} \\ \overset{3}{\sqrt{3}/4} & \overset{4}{-3/4} & \overset{1}{-\sqrt{3}/4} & \overset{2}{3/4} \end{bmatrix}$$

$$[K^3] = \frac{AE}{L} \begin{bmatrix} \overset{5}{1} & \overset{6}{0} & \overset{3}{-1} & \overset{4}{0} \\ \overset{5}{0} & \overset{6}{0} & \overset{3}{0} & \overset{4}{0} \\ \overset{3}{-1} & \overset{4}{0} & \overset{5}{1} & \overset{6}{0} \\ \overset{3}{0} & \overset{4}{0} & \overset{5}{0} & \overset{6}{0} \end{bmatrix}$$

$$[K] = \frac{AE}{L} \begin{bmatrix} \overset{1}{1/2} & \overset{2}{0} & \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{-1/4} & \overset{6}{-\sqrt{3}/4} \\ 0 & \overset{2}{3/2} & \overset{3}{\sqrt{3}/4} & \overset{4}{-3/4} & \overset{5}{-\sqrt{3}/4} & \overset{6}{-3/4} \\ \text{---} & \text{---} & \overset{3}{-1/4} & \overset{4}{\sqrt{3}/4} & \overset{5}{5/4} & \overset{6}{-\sqrt{3}/4} \\ \overset{3}{\sqrt{3}/4} & \overset{4}{-3/4} & \text{---} & \text{---} & \overset{5}{0} & \overset{6}{0} \\ \text{---} & \text{---} & \text{---} & \text{---} & \overset{5}{-1/4} & \overset{6}{-\sqrt{3}/4} \\ \overset{3}{-\sqrt{3}/4} & \overset{4}{-3/4} & \overset{5}{0} & \overset{6}{0} & \text{---} & \text{---} \end{bmatrix}$$



**THANK
YOU !**



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

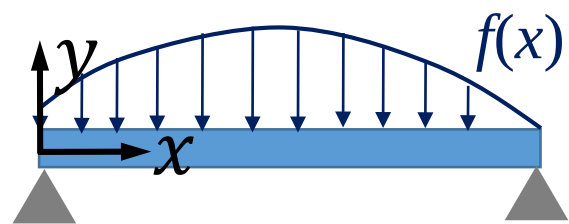
Week 6: Finite element formulation for plane trusses and frames

Lecture 30: Analysis of Euler-Bernoulli Beam (Computer Implementation)

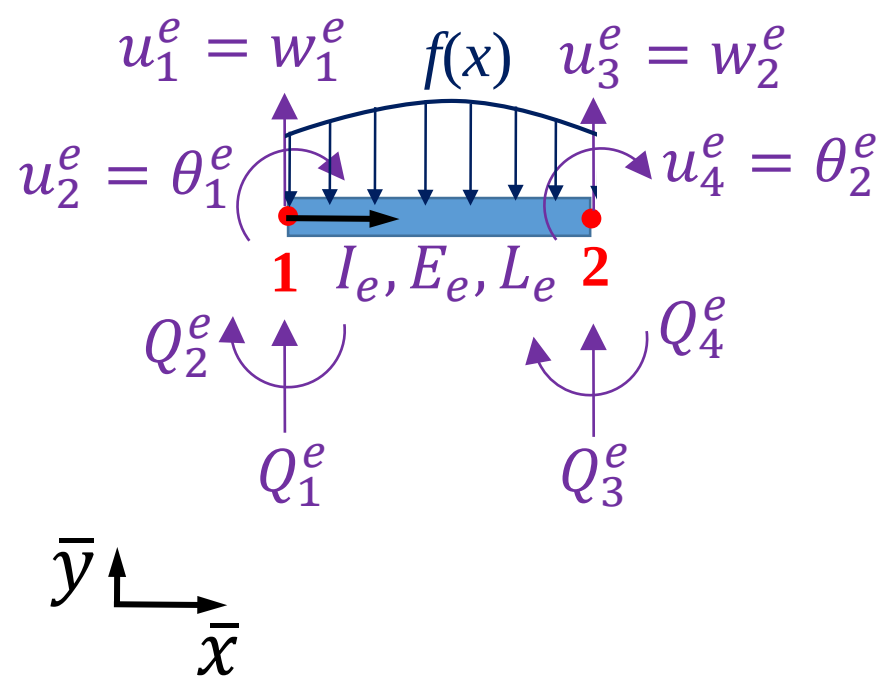
Biswanath Banerjee
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Amit Shaw
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Recall: Element Formulation (Euler-Bernoulli Beam)



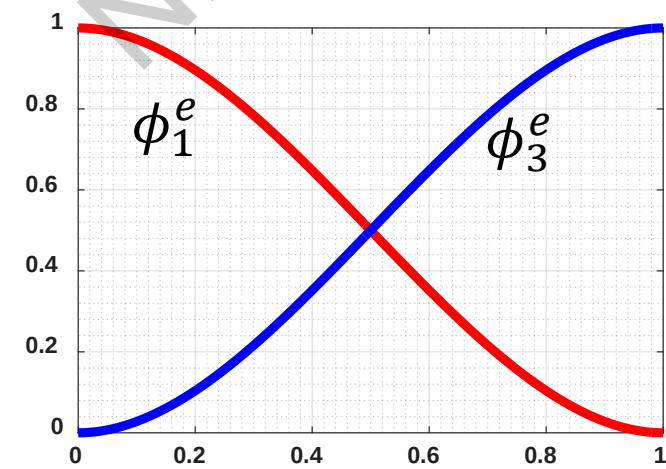
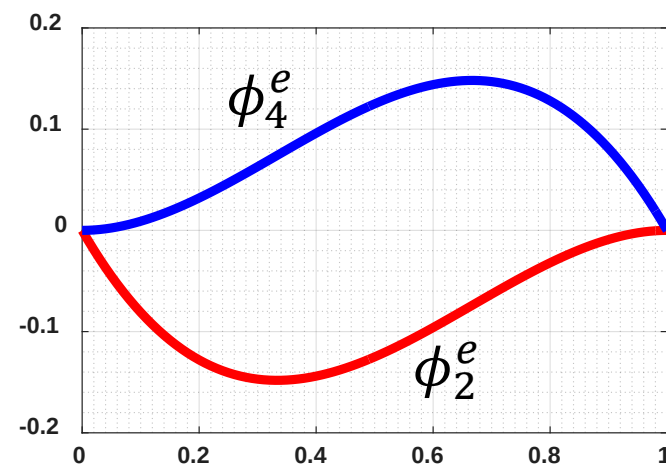
Element: Building block



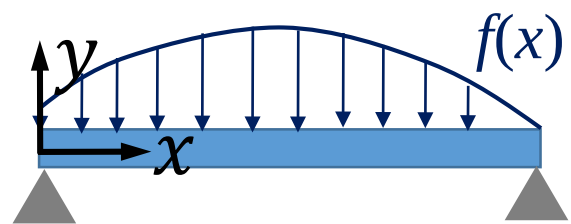
Shape Functions

$$\phi_1^e = 1 - 3 \left(\frac{\bar{x}}{L_e} \right)^2 + 2 \left(\frac{\bar{x}}{L_e} \right)^3 \qquad \phi_3^e = 3 \left(\frac{\bar{x}}{L_e} \right)^2 - 2 \left(\frac{\bar{x}}{L_e} \right)^3$$

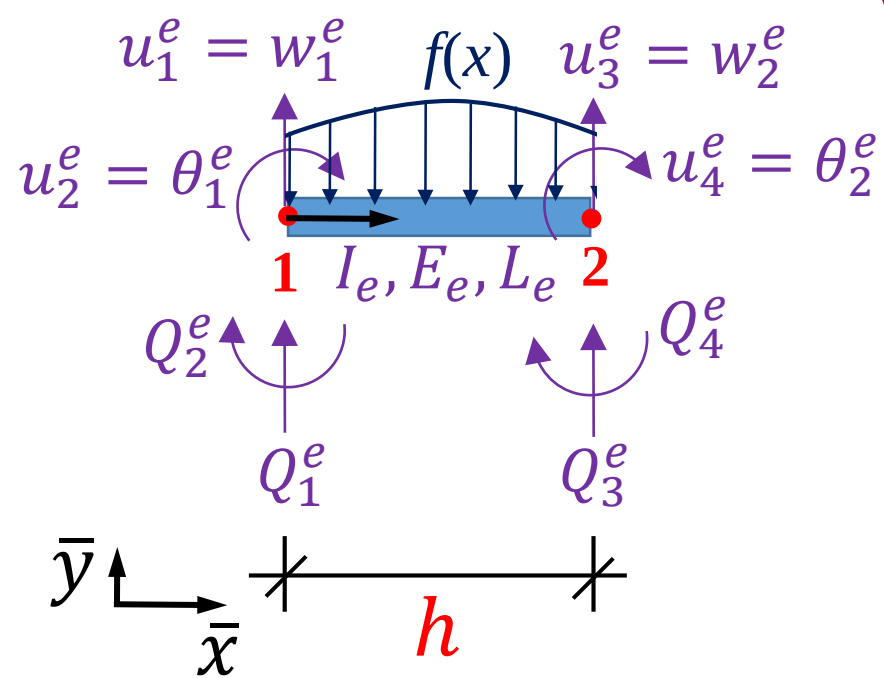
$$\phi_2^e = -\bar{x} \left(1 - \frac{\bar{x}}{L_e} \right)^2 \qquad \phi_4^e = -\bar{x} \left[\left(\frac{\bar{x}}{L_e} \right)^2 - \frac{\bar{x}}{L_e} \right]$$



Recall: Element Formulation (Euler-Bernoulli Beam)



Element: Building block



Element Equation

$$\frac{2E_e I_e}{L_e^3} \begin{bmatrix} 6 & -3L_e & -6 & -3L_e \\ -3L_e & 2L_e^2 & 3L_e & L_e^2 \\ -6 & 3L_e & 6 & 3L_e \\ -3L_e & L_e^2 & 3L_e & 2L_e^2 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

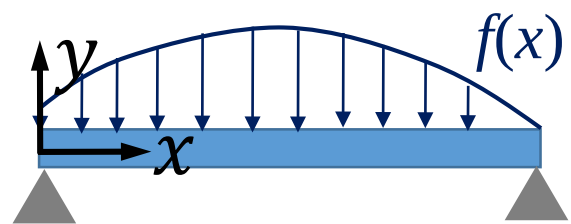
Element Stiffness Matrix

Element Displacement Vector

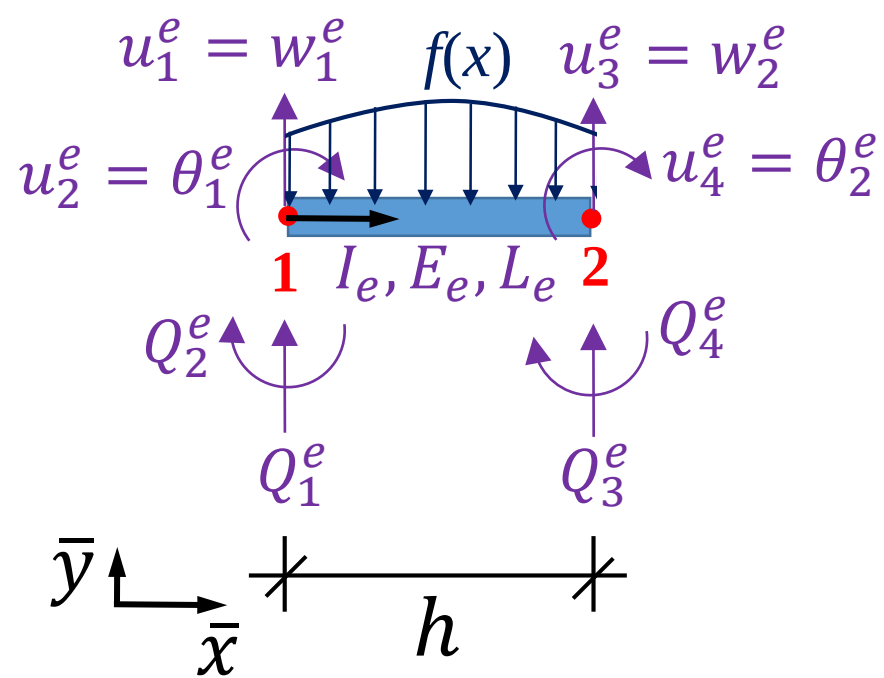
Element Load Vector

$$f_i^e = \int_{x_1}^{x_2} \phi_i^e f(x) dx = \int_0^{L_e} \phi_i^e(\bar{x}) f(\bar{x}) d\bar{x}$$

Recall: Element Formulation (Euler-Bernoulli Beam)



Element: Building block



Element Equation

1

2

$$\frac{2E_e I_e}{h^3} \begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

1

2

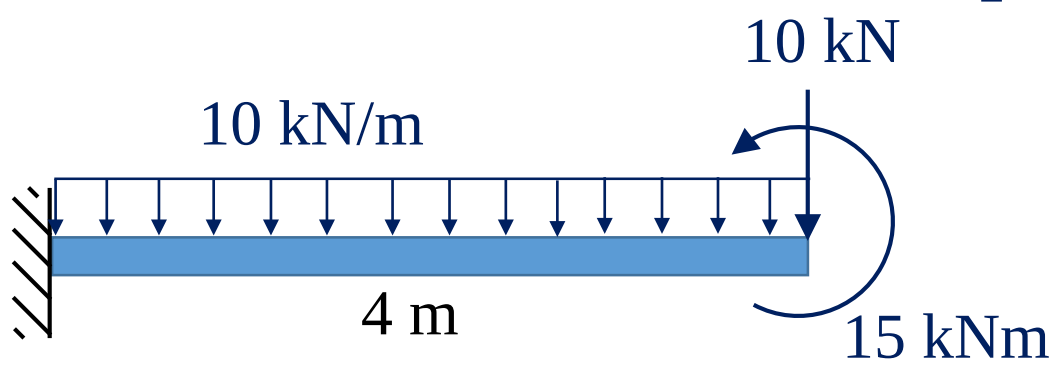
Element Stiffness Matrix

Element Displacement Vector

Element Load Vector

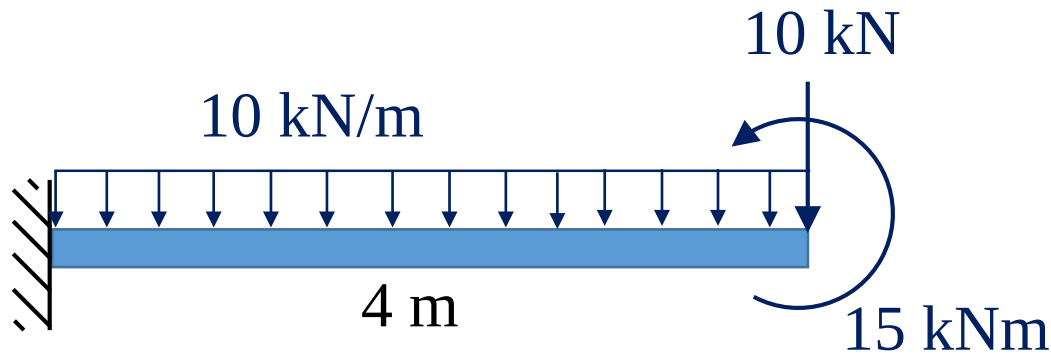
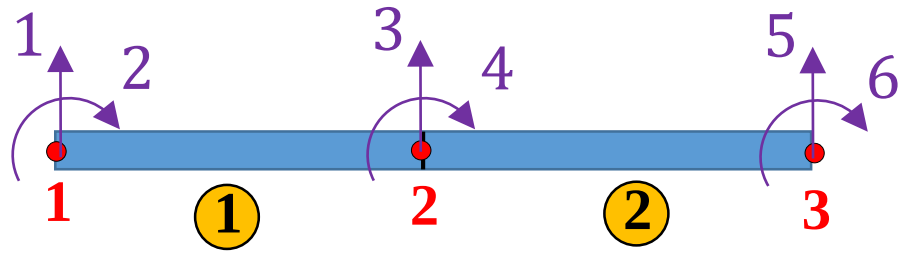
$$f_i^e = \int_{x_1}^{x_2} \phi_i^e f(x) dx = \int_0^h \phi_i^e(\bar{x}) f(\bar{x}) d\bar{x}$$

Euler-Bernoulli Beam: Example



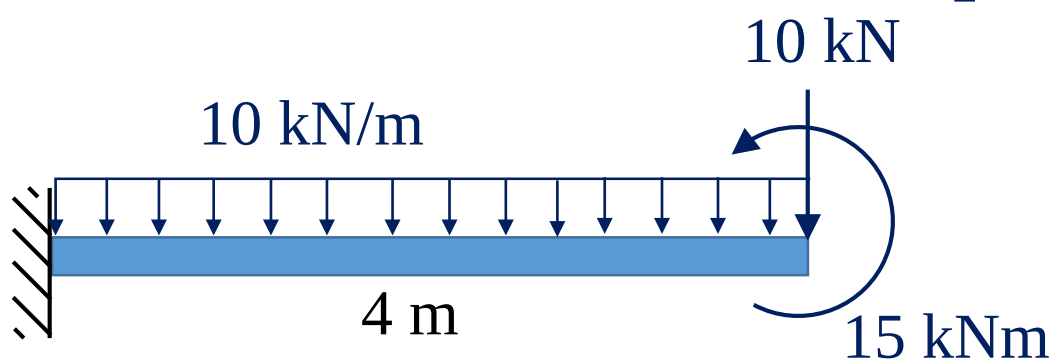
NPTEL

Euler-Bernoulli Beam Example: Discretization

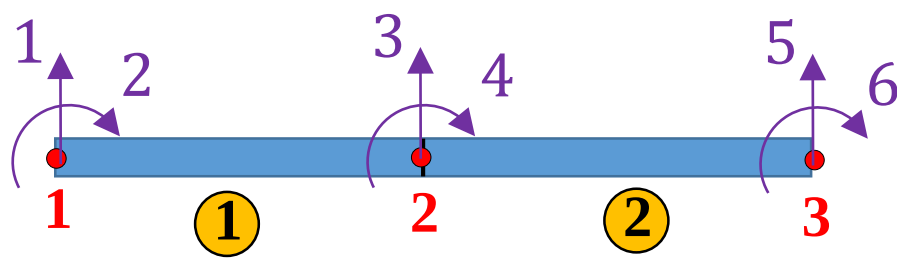


Element Number	Global nodes of the element (1 – 2)	Geometric and Material properties
1	1 – 2	$E, I, h = 2$
2	2 – 3	$E, I, h = 2$

Euler-Bernoulli Beam Example: Element Formulation



$$K^e = \frac{2E_e I_e}{h^3} \begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$

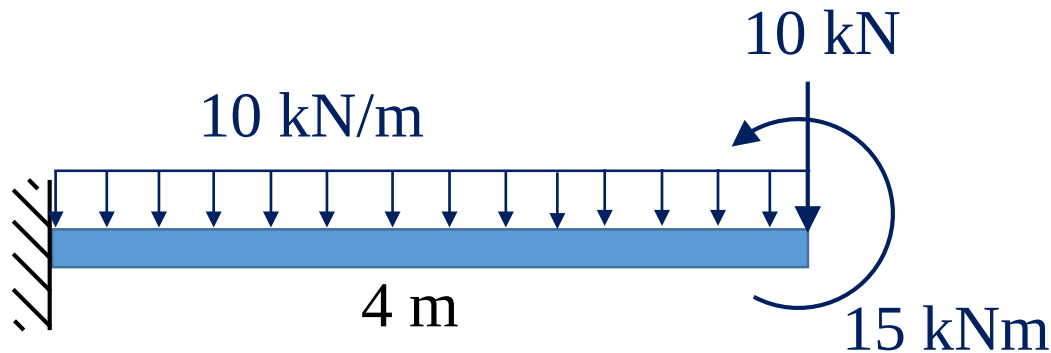
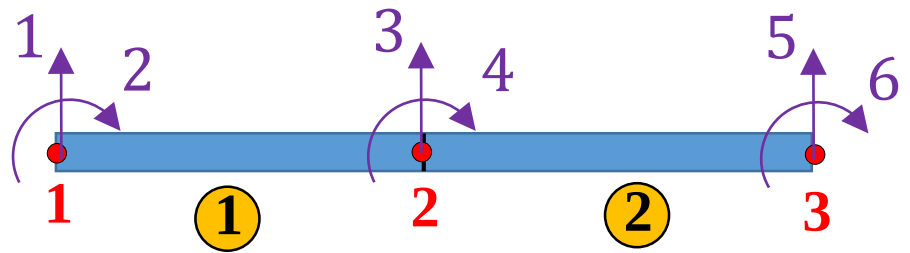


Element Number	Global nodes of the element (1 – 2)	Geometric and Material properties
1	1 – 2	$E, I, h = 2$
2	2 – 3	$E, I, h = 2$

$$K^1 = EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 \\ -1.5 & 2 & 1.5 & 1 \\ -1.5 & 1.5 & 1.5 & 1.5 \\ -1.5 & 1 & 1.5 & 2 \end{bmatrix}$$

$$K^2 = EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 \\ -1.5 & 2 & 1.5 & 1 \\ -1.5 & 1.5 & 1.5 & 1.5 \\ -1.5 & 1 & 1.5 & 2 \end{bmatrix}$$

Euler-Bernoulli Beam Example: **Element Formulation**



$$\begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$f_i^e = \int_0^h \phi_i^e(\bar{x}) f(\bar{x}) d\bar{x}$$

For $f(\bar{x}) = q$, $\begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix} = \frac{qh}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$

$q = -10 \text{ kN/m}$ and $h = 2\text{m}$

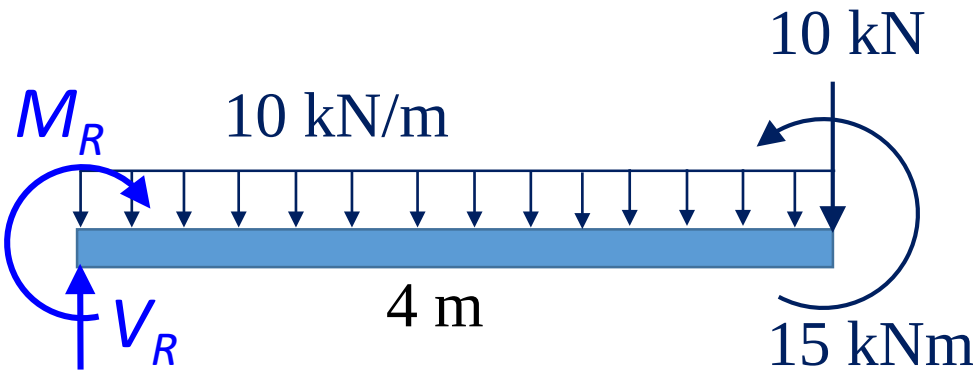
$$\begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

1 2 3 4

$$\begin{Bmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

3 4 5 6

Euler-Bernoulli Beam Example: Assembly (Global Equation)

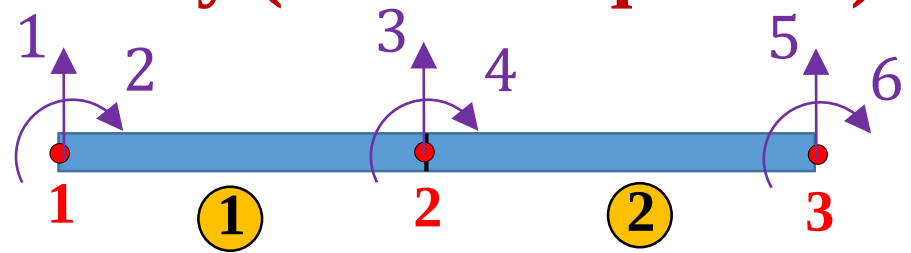


$$\begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

1 2 3 4

$$\begin{Bmatrix} F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

3 4 5 6



$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix} + \begin{Bmatrix} -Q_1^1 \\ -Q_2^1 \\ -Q_3^1 + Q_1^2 \\ -Q_4^1 + Q_2^2 \\ -Q_3^2 \\ -Q_4^2 \end{Bmatrix}$$

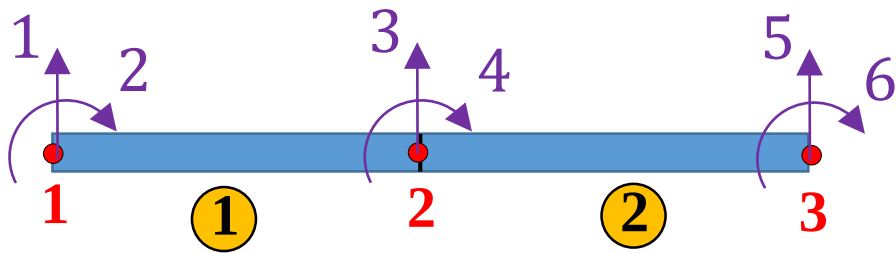
V_R
 M_R
0
0
-10
-15

$$\Rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -60 \\ 0 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} V_R \\ M_R \\ 0 \\ 0 \\ -10 \\ -15 \end{Bmatrix}$$

Euler-Bernoulli Beam Example: Assembly (Global Equation)

$$K^1 = EI \begin{bmatrix} \overset{1}{1.5} & \overset{2}{-1.5} & \overset{3}{-1.5} & \overset{4}{-1.5} \\ -1.5 & 2 & 1.5 & 1 \\ -1.5 & 1.5 & 1.5 & 1.5 \\ -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{matrix} \overset{1}{1} \\ \overset{2}{2} \\ \overset{3}{3} \\ \overset{4}{4} \end{matrix}$$

$$K^2 = EI \begin{bmatrix} \overset{3}{1.5} & \overset{4}{-1.5} & \overset{5}{-1.5} & \overset{6}{-1.5} \\ -1.5 & 2 & 1.5 & 1 \\ -1.5 & 1.5 & 1.5 & 1.5 \\ -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{matrix} \overset{3}{3} \\ \overset{4}{4} \\ \overset{5}{5} \\ \overset{6}{6} \end{matrix}$$



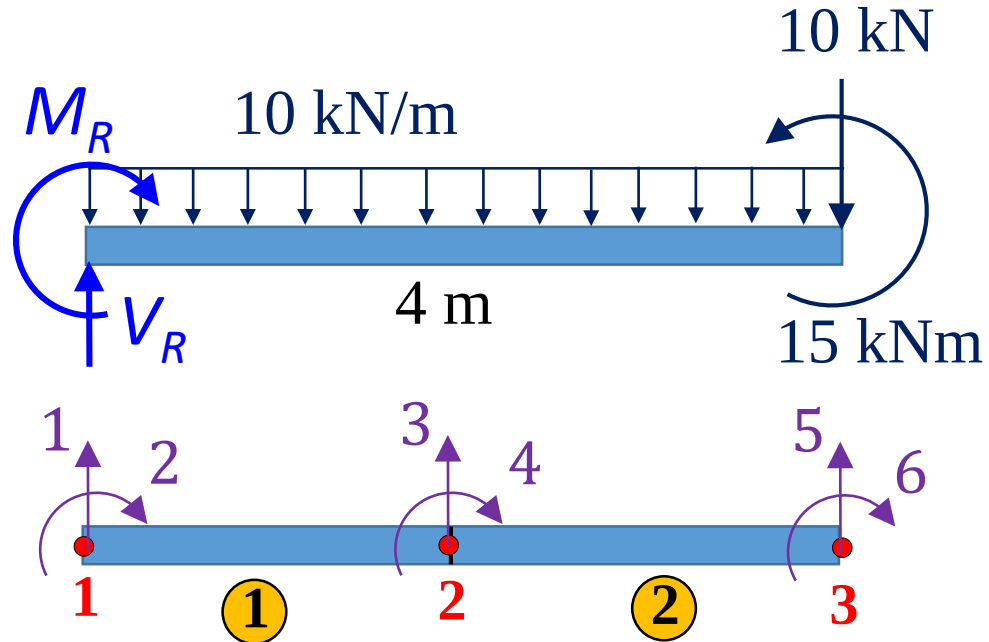
NPTTEL

$$K = EI \begin{bmatrix} \overset{1}{1.5} & \overset{2}{-1.5} & \overset{3}{-1.5} & \overset{4}{-1.5} & \overset{5}{0} & \overset{6}{0} \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix}$$

K^1 (upward arrow at Node 1)
 K^2 (downward arrow at Node 3)

Euler-Bernoulli Beam Example: **Assembly (Global Equation)**

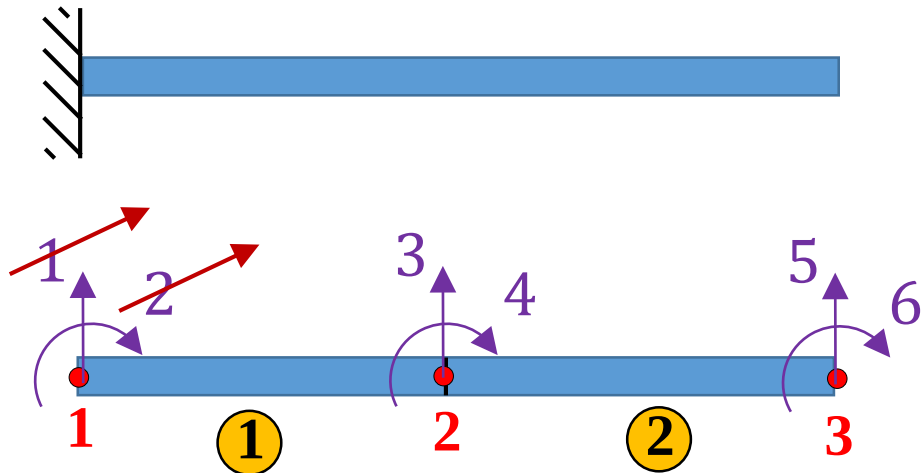
$$EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 & 0 & 0 \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -60 \\ 0 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} V_R \\ M_R \\ 0 \\ 0 \\ -10 \\ -15 \end{Bmatrix}$$



Euler-Bernoulli Beam Example: **Boundary Condition**

$$EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 & 0 & 0 \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -60 \\ 0 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} V_R \\ M_R \\ 0 \\ 0 \\ -10 \\ -15 \end{Bmatrix}$$

$\begin{matrix} \nearrow 0 \\ \nearrow 0 \end{matrix}$



Euler-Bernoulli Beam Example: **Boundary Condition**

$$EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 & 0 & 0 \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 \\ 10 \\ -60 \\ 0 \\ -30 \\ -10 \end{Bmatrix} + \begin{Bmatrix} V_R \\ M_R \\ 0 \\ 0 \\ -10 \\ -15 \end{Bmatrix}$$

Unknown Displacements

u_3, u_4, u_5, u_6

Unknown Forces

V, M

Euler-Bernoulli Beam Example: **Solution**

$$EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 & 0 & 0 \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 + 3V_R \\ 10 + 3M_R \\ -60 \\ 0 \\ -60 \\ -55 \end{Bmatrix}$$

$$\begin{bmatrix} K_{uk} & K_{uu} \\ K_{kk} & K_{ku} \end{bmatrix} \begin{Bmatrix} u_k \\ u_u \end{Bmatrix} = \begin{Bmatrix} F_u \\ F_k \end{Bmatrix}$$

$$\Rightarrow \begin{cases} [K_{ku}]\{u_u\} + [K_{kk}]\{u_k\} = \{F_k\} \\ [K_{uu}]\{u_u\} + [K_{uk}]\{u_k\} = \{F_u\} \end{cases}$$

Determine
Unknown
Displacement u_u

Determine
Unknown
Forces F_u

Euler-Bernoulli Beam Example: **Solution**

$$EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 & 0 & 0 \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 + 3V_R \\ 10 + 3M_R \\ -60 \\ 0 \\ -60 \\ -55 \end{Bmatrix}$$

$$\Rightarrow EI \begin{bmatrix} 3 & 0 & -1.5 & -1.5 \\ 0 & 4 & 1.5 & 1 \\ -1.5 & 1.5 & 1.5 & 1.5 \\ -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -60 \\ 0 \\ -60 \\ -55 \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3EI} \begin{Bmatrix} -450 \\ 370 \\ -1240 \\ 380 \end{Bmatrix}$$

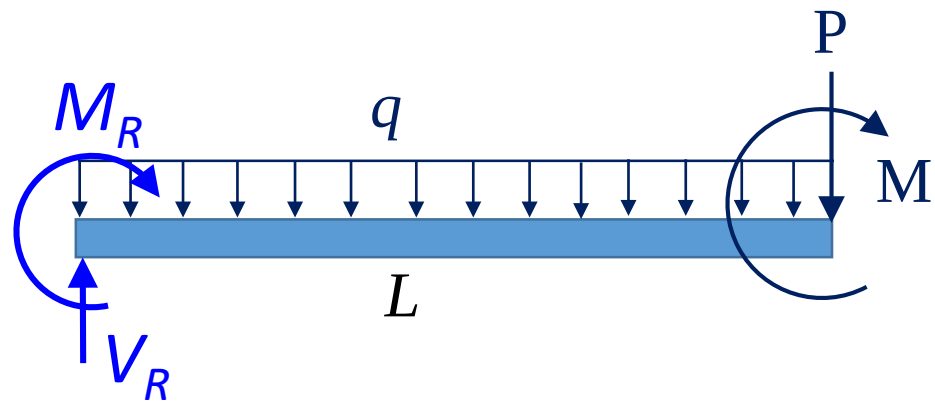
Euler-Bernoulli Beam Example: **Solution**

$$EI \begin{bmatrix} 1.5 & -1.5 & -1.5 & -1.5 & 0 & 0 \\ -1.5 & 2 & 1.5 & 1 & 0 & 0 \\ -1.5 & 1.5 & 3 & 0 & -1.5 & -1.5 \\ -1.5 & 1 & 0 & 4 & 1.5 & 1 \\ 0 & 0 & -1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & 1 & 1.5 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 + 3V_R \\ 10 + 3M_R \\ -60 \\ 0 \\ -60 \\ -55 \end{Bmatrix}$$

$$\Rightarrow EI \begin{bmatrix} -1.5 & -1.5 & 0 & 0 \\ 1.5 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 + 3V_R \\ 10 + 3M_R \end{Bmatrix}$$

$$\Rightarrow EI \begin{bmatrix} -1.5 & -1.5 & 0 & 0 \\ 1.5 & 1 & 0 & 0 \end{bmatrix} \frac{1}{3EI} \begin{Bmatrix} -450 \\ 370 \\ -1240 \\ 380 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} -30 + 3V_R \\ 10 + 3M_R \end{Bmatrix} \Rightarrow \begin{Bmatrix} V_R \\ M_R \end{Bmatrix} = \begin{Bmatrix} 50 \\ 105 \end{Bmatrix}$$

Euler-Bernoulli Beam Example: **Solution**



$q = -10 \text{ kN/m}$
 $P = -10 \text{ kN}$
 $M = -15 \text{ kNm}$
 $L = 4$
 $EI = \text{Constant}$

$$w = \frac{Px^2}{6EI}(3L - x) + \frac{qx^2}{24EI}(x^2 + 6L^2 - 4Lx) + \frac{Mx^2}{2EI}$$

$$\theta = \frac{dw}{dx}$$

Analytical
(Strength of Materials)

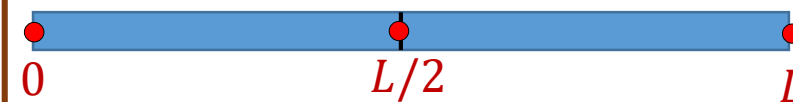
$$V_R = 50 \text{ kN}$$

$$M_R = 105 \text{ kNm}$$

$$w_{L/2} = -\frac{450}{3EI} \quad \theta_{L/2} = \frac{380}{3EI}$$

$$w_L = -\frac{1240}{3EI} \quad \theta_{L/2} = \frac{370}{3EI}$$

FEM



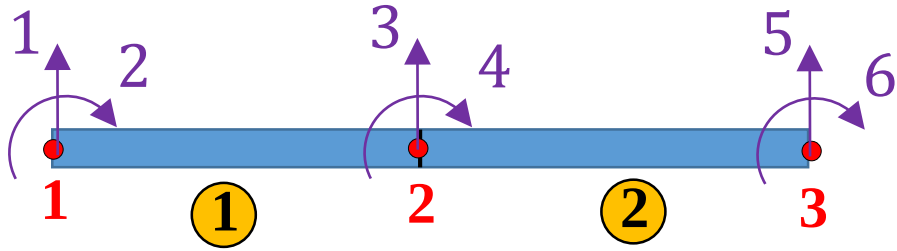
$$V_R = 50 \text{ kN}$$

$$M_R = 105 \text{ kNm}$$

$$w_{L/2} = -\frac{450}{3EI} \quad \theta_{L/2} = \frac{380}{3EI}$$

$$w_L = -\frac{1240}{3EI} \quad \theta_{L/2} = \frac{370}{3EI}$$

Euler-Bernoulli Beam Example: **Post-processing**



$$w^e(\bar{x}) = \sum_{j=1}^4 \phi_j^e u_j^e$$

$$\begin{aligned} \Rightarrow w^1(\bar{x}) \\ &= u_1 \phi_1^1(\bar{x}) + u_2 \phi_2^1(\bar{x}) \\ &\quad + u_3 \phi_3^1(\bar{x}) + u_4 \phi_4^1(\bar{x}) \end{aligned}$$

$$\begin{aligned} \Rightarrow w^2(\bar{x}) \\ &= u_3 \phi_1^2(\bar{x}) + u_4 \phi_2^2(\bar{x}) \\ &\quad + u_5 \phi_3^2(\bar{x}) + u_6 \phi_4^2(\bar{x}) \end{aligned}$$

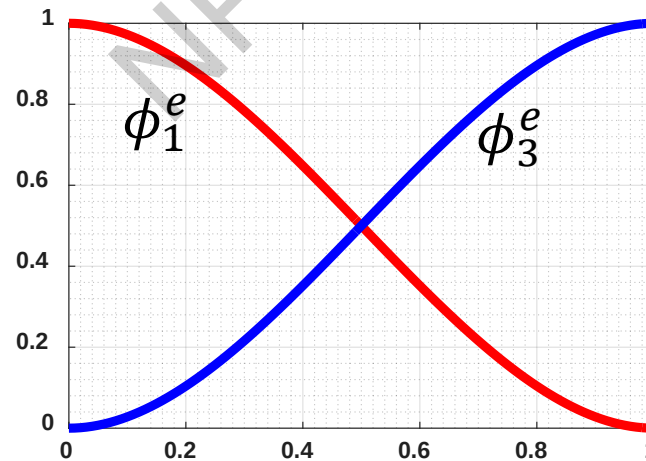
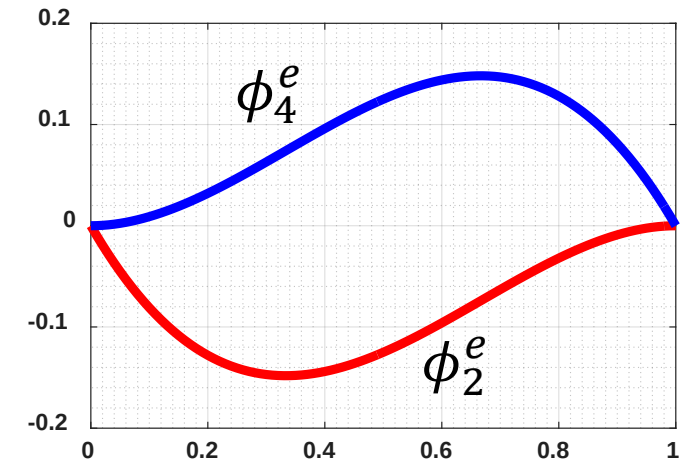
Shape Functions

$$\phi_1^e = 1 - 3 \left(\frac{\bar{x}}{h} \right)^2 + 2 \left(\frac{\bar{x}}{h} \right)^3$$

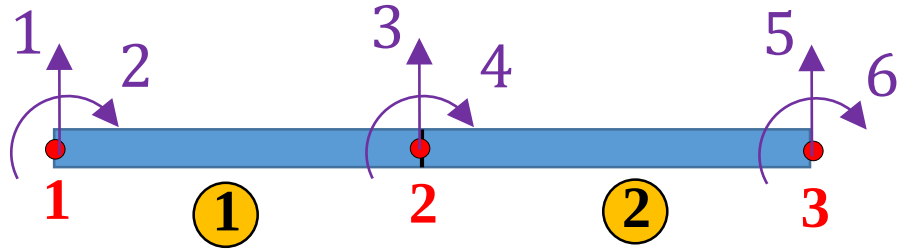
$$\phi_3^e = 3 \left(\frac{\bar{x}}{h} \right)^2 - 2 \left(\frac{\bar{x}}{h} \right)^3$$

$$\phi_2^e = -\bar{x} \left(1 - \frac{\bar{x}}{h} \right)^2$$

$$\phi_4^e = -\bar{x} \left[\left(\frac{\bar{x}}{h} \right)^2 - \frac{\bar{x}}{h} \right]$$



Euler-Bernoulli Beam Example: **Post-processing**



Shape Functions

$$\phi_1^e = 1 - 3 \left(\frac{\bar{x}}{h} \right)^2 + 2 \left(\frac{\bar{x}}{h} \right)^3$$

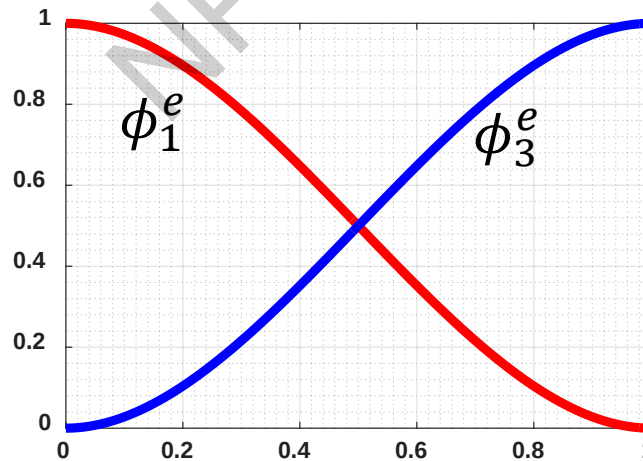
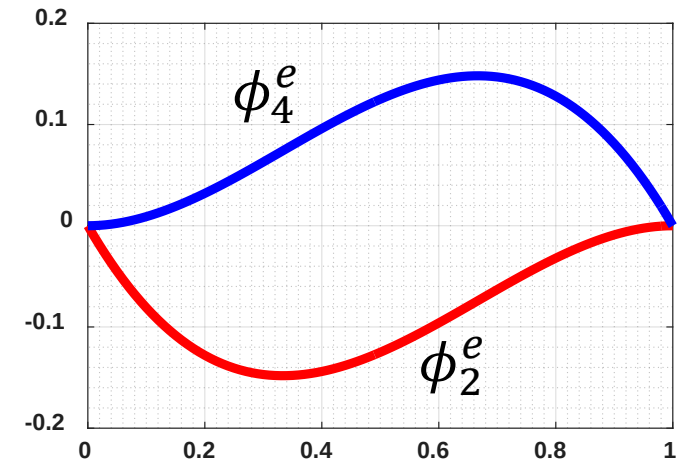
$$\phi_3^e = 3 \left(\frac{\bar{x}}{h} \right)^2 - 2 \left(\frac{\bar{x}}{h} \right)^3$$

$$\phi_2^e = -\bar{x} \left(1 - \frac{\bar{x}}{h} \right)^2$$

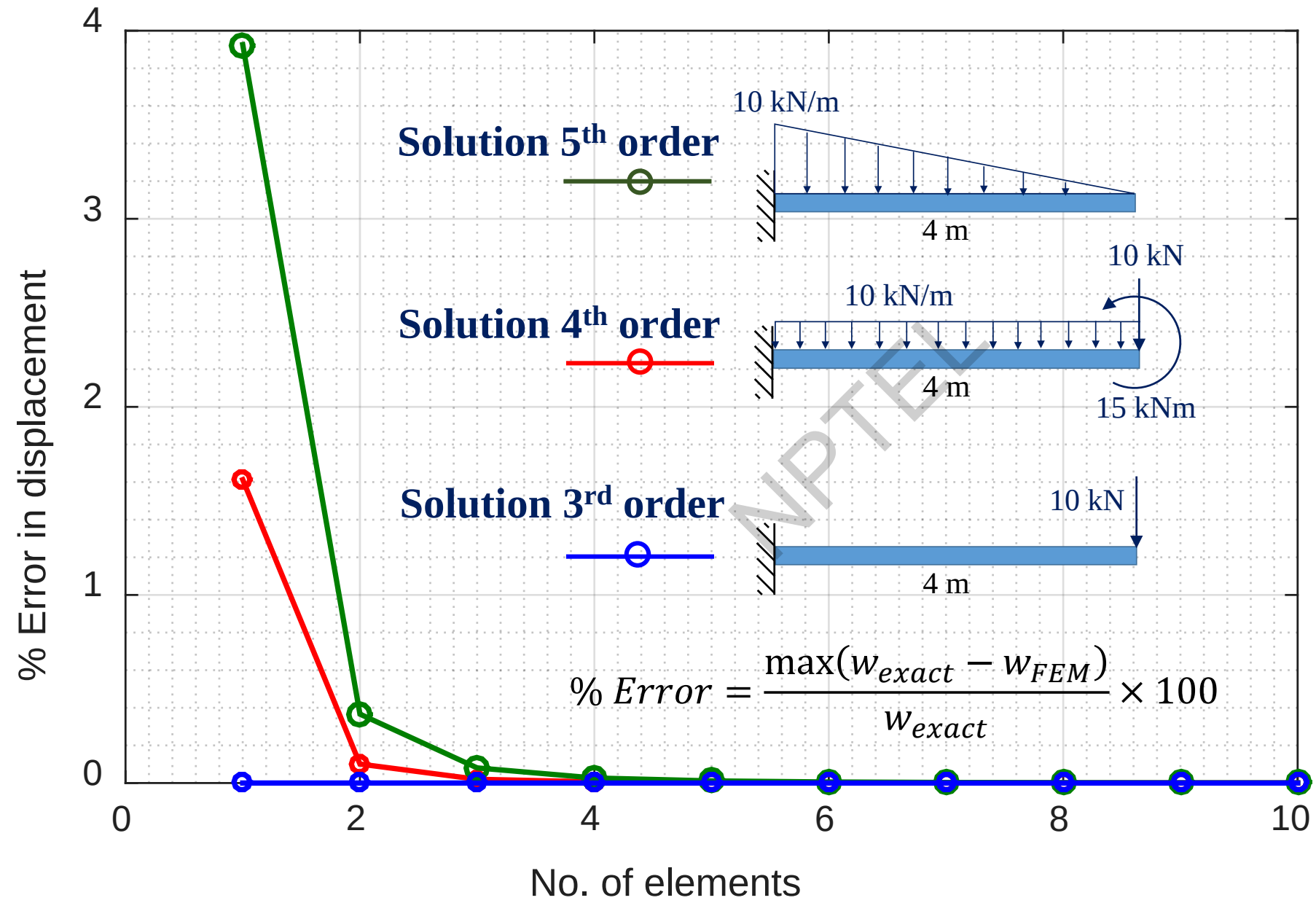
$$\phi_4^e = -\bar{x} \left[\left(\frac{\bar{x}}{h} \right)^2 - \frac{\bar{x}}{h} \right]$$

$$M(\bar{x}) = EI \frac{d^2 w(\bar{x})}{d\bar{x}^2}$$

$$\Rightarrow M(\bar{x}) = EI \sum_{j=1}^4 \frac{d^2 \phi_j^e}{d\bar{x}^2} u_j^e$$



Euler-Bernoulli Beam Example: **Convergence**





**THANK
YOU !**



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

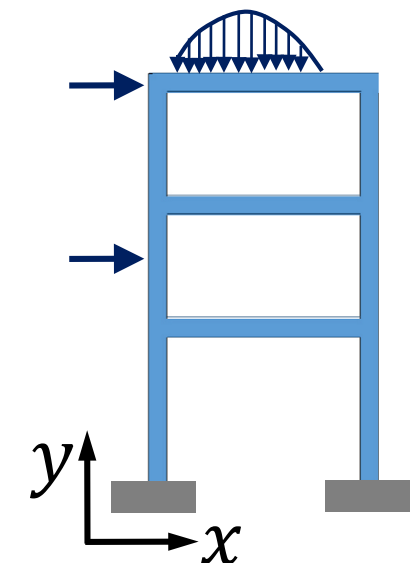
Week 6: Finite element formulation for plane trusses and frames

Lecture 31: Analysis of Plane Frame (Computer Implementation)

Biswanath Banerjee
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Amit Shaw
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Recall: Element Formulation



1

2

3

4

5

6

1

2

3

4

5

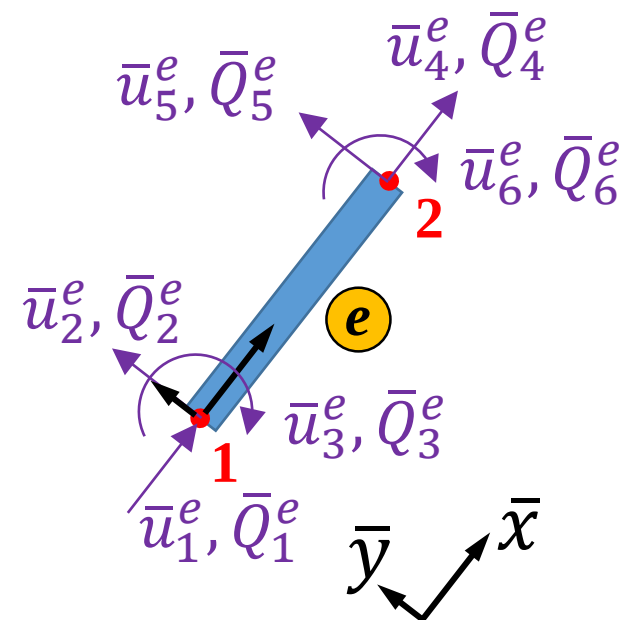
6

1

2

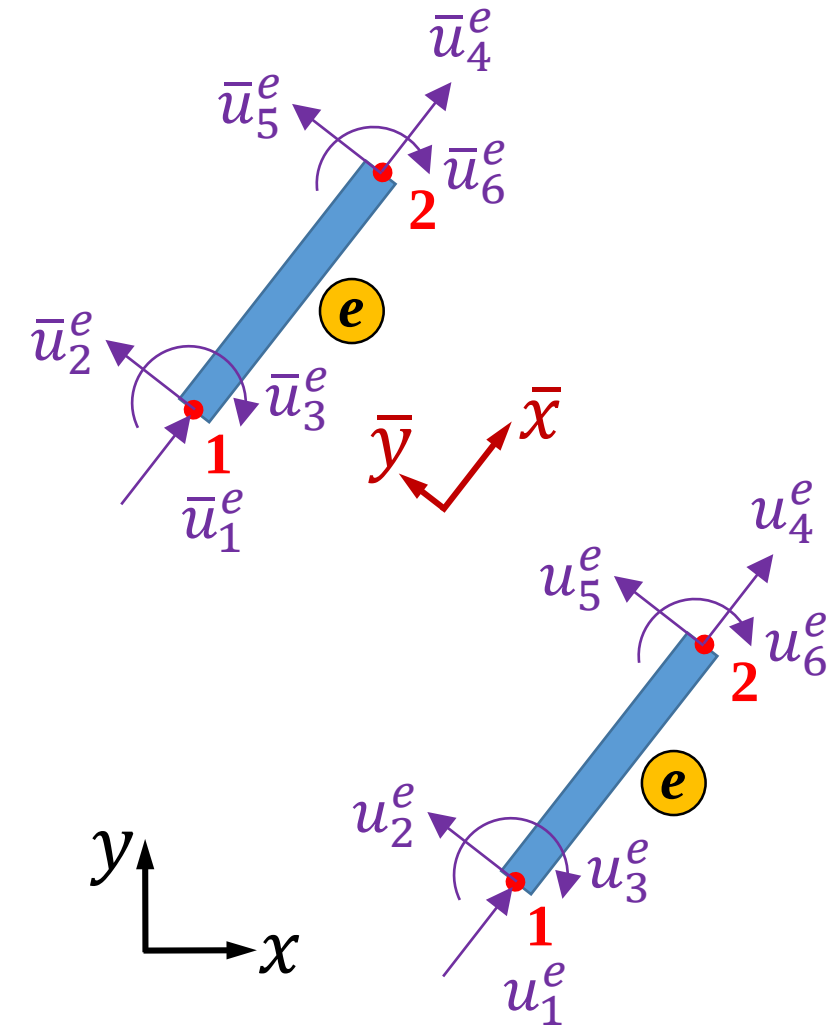
$$[\bar{K}^e] = \frac{2E_e I_e}{h^3} \begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3h & 0 & -6 & -3h \\ 0 & -3h & 2h^2 & 0 & 3h & h^2 \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3h & 0 & 6 & 3h \\ 0 & -3h & h^2 & 0 & 3h & 2h^2 \end{bmatrix}$$

$$\mu = \frac{Ah^2}{2I_e}$$



$$[\bar{K}^e]\{\bar{u}^e\} = \{\bar{F}^e\}$$

Recall: Element Formulation



$$[T^e] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{1} \\ \text{2} \\ \text{1} \\ \text{2} \end{matrix}$$

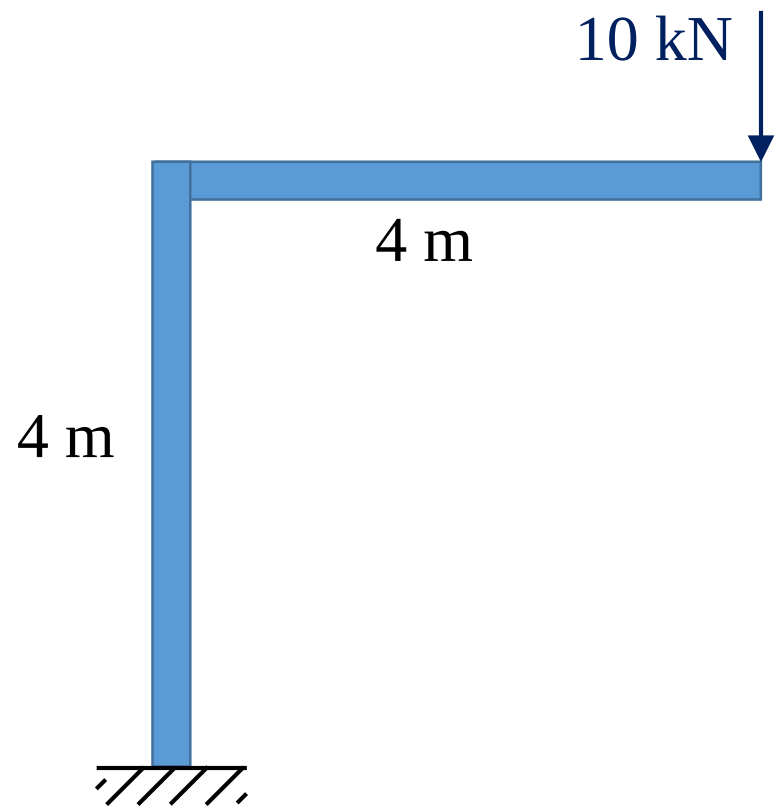
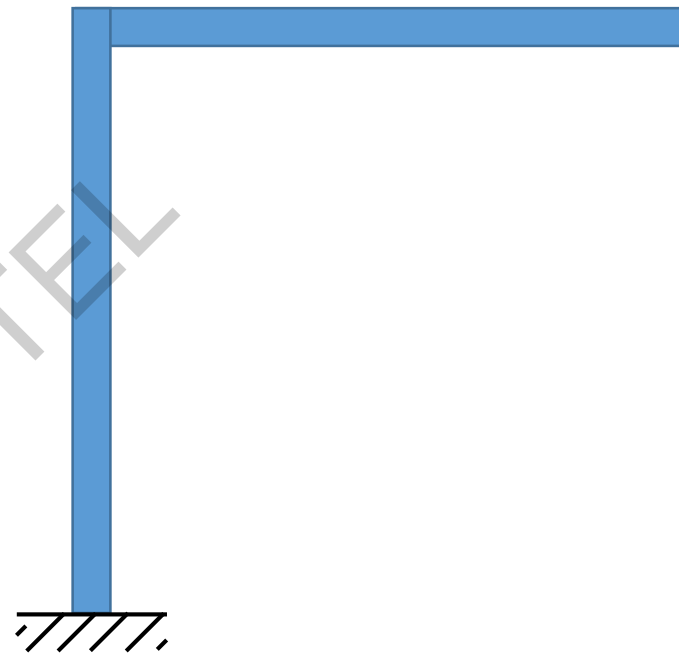
$$\{\bar{u}^e\} = [T^e]\{u^e\}$$

$$\{\bar{F}^e\} = [T^e]\{F^e\}$$

$$[K^e] = [T^e]^T [\bar{K}^e] [T^e]$$

$$[K^e]\{u^e\} = \{F^e\}$$

Plane Frame: Example



NPTEL

GIGO



Garbage In, Garbage Out